Math 3450

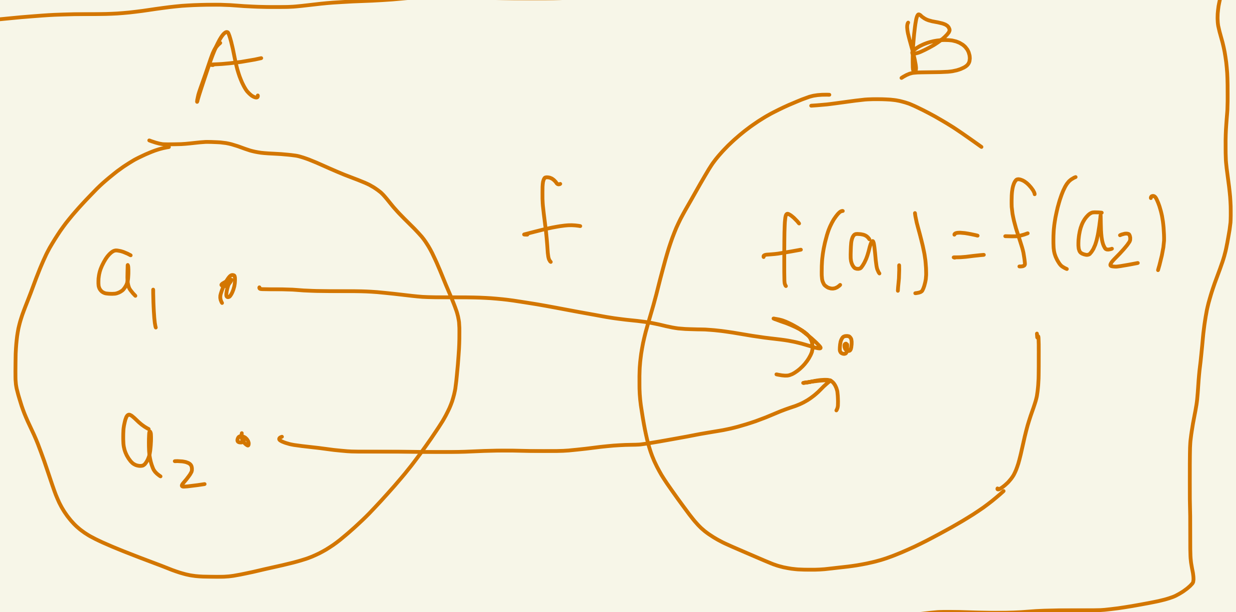
3/19/24

<u>Def:</u> Let A and B be sets.

Let $f: A \to B$ be a function. We say that f is <u>injective</u> or <u>one-to-one</u> if the following is true:

For all $a_1, a_2 \in A$, if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$

Ie you <u>cannot</u> have this



Another way to define:
For all $a_1, a_2 \in A$,
if $f(a_1) = f(a_2)$, then $a_1 = a_2$

How to prove $f: A \to B$ is one-to-one

Let $a_1, a_2 \in A$.
Suppose $f(a_1) = f(a_2)$

$\vdots$ (proof stuff)

conclude $a_1 = a_2$

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = -4x+5$. Let's prove f is one-to-one.

pf: Suppose $x_1, x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$.

Then, $-4x_1 + 5 = -4x_2 + 5$.

-5

Thus, $-4x_1 = -4x_2$.

$\times\left(-\frac{1}{4}\right)$

So, $x_1 = x_2$.

Thus, f is one-to-one

$\blacksquare$

How to show $f: A \to B$ is not one-to-one

Find specific $x_1, x_2 \in A$ where $x_1 \neq x_2$ but $f(x_1) = f(x_2)$

Ex: Let $n \in \mathbb{Z}$, $n \geq 2$.

Define $f: \mathbb{Z}_n \to \mathbb{Z}_n$ by $f(\overline{x}) = (\overline{x})^2$.

Claim: f is well-defined.

pf of claim:

① Given $\overline{x} \in \mathbb{Z}_n$ where $x \in \mathbb{Z}$ we have that

$$f(\overline{x}) = \overline{x}^2 = \overline{x} \cdot \overline{x} = \overline{x^2}.$$

Since $x \in \mathbb{Z}$ we know $x^2 \in \mathbb{Z}$. Thus, $f(\overline{x}) = \overline{x^2} \in \mathbb{Z}_n$.

② Suppose $\overline{x_1}, \overline{x_2} \in \mathbb{Z}_n$ and $\overline{x_1} = \overline{x_2}$. Then,

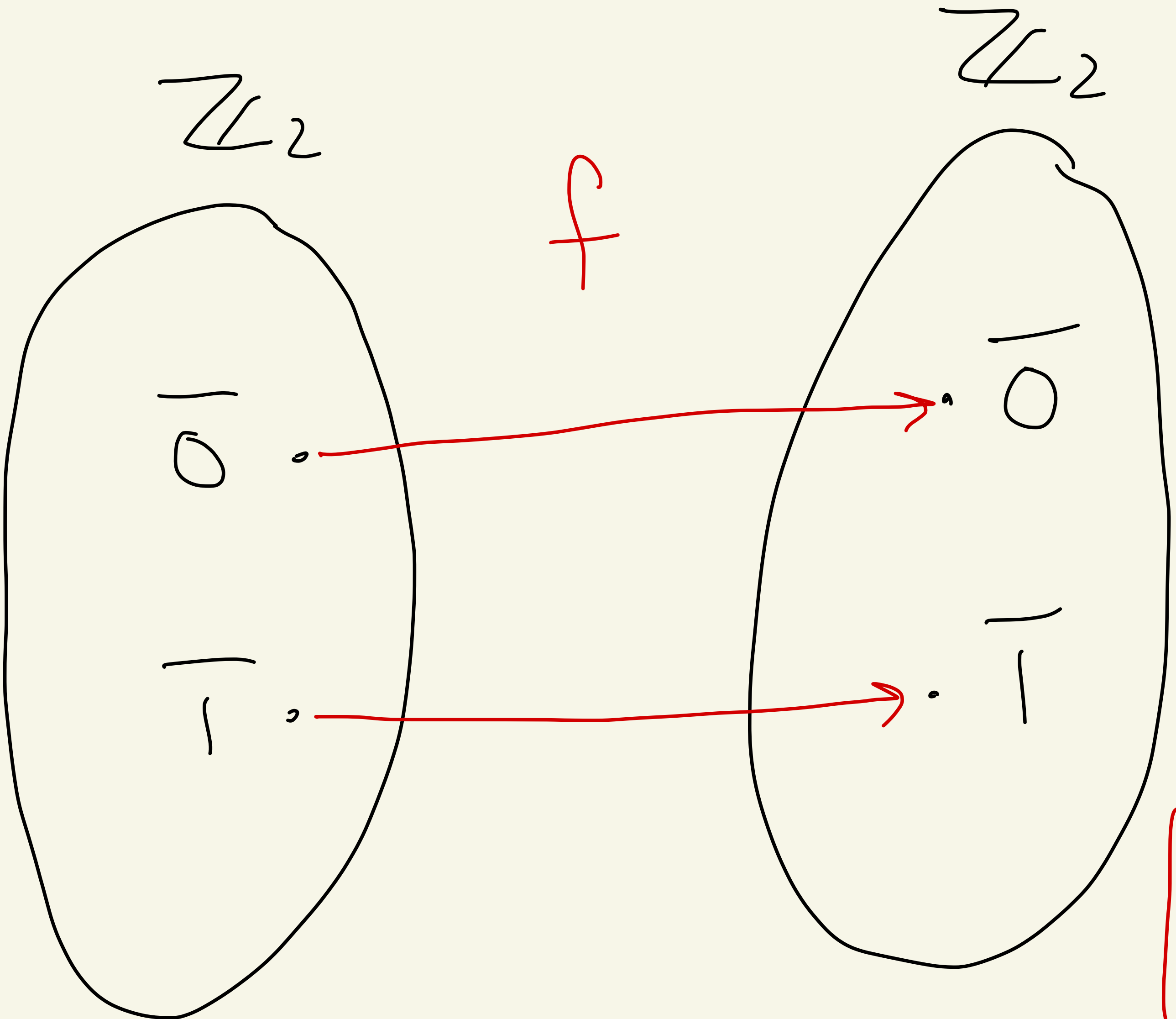
$$f(\overline{x_1}) = \overline{x_1}^2 = \overline{x_2}^2 = f(\overline{x_2})$$

mult. is well-defined in $\mathbb{Z}_n$, if $\overline{a} = \overline{c}$ and $\overline{b} = \overline{d}$, then $\overline{a}\,\overline{b} = \overline{c}\,\overline{d}$

Use with $\overline{a} = \overline{b} = \overline{x_1}$ and $\overline{c} = \overline{d} = \overline{x_2}$

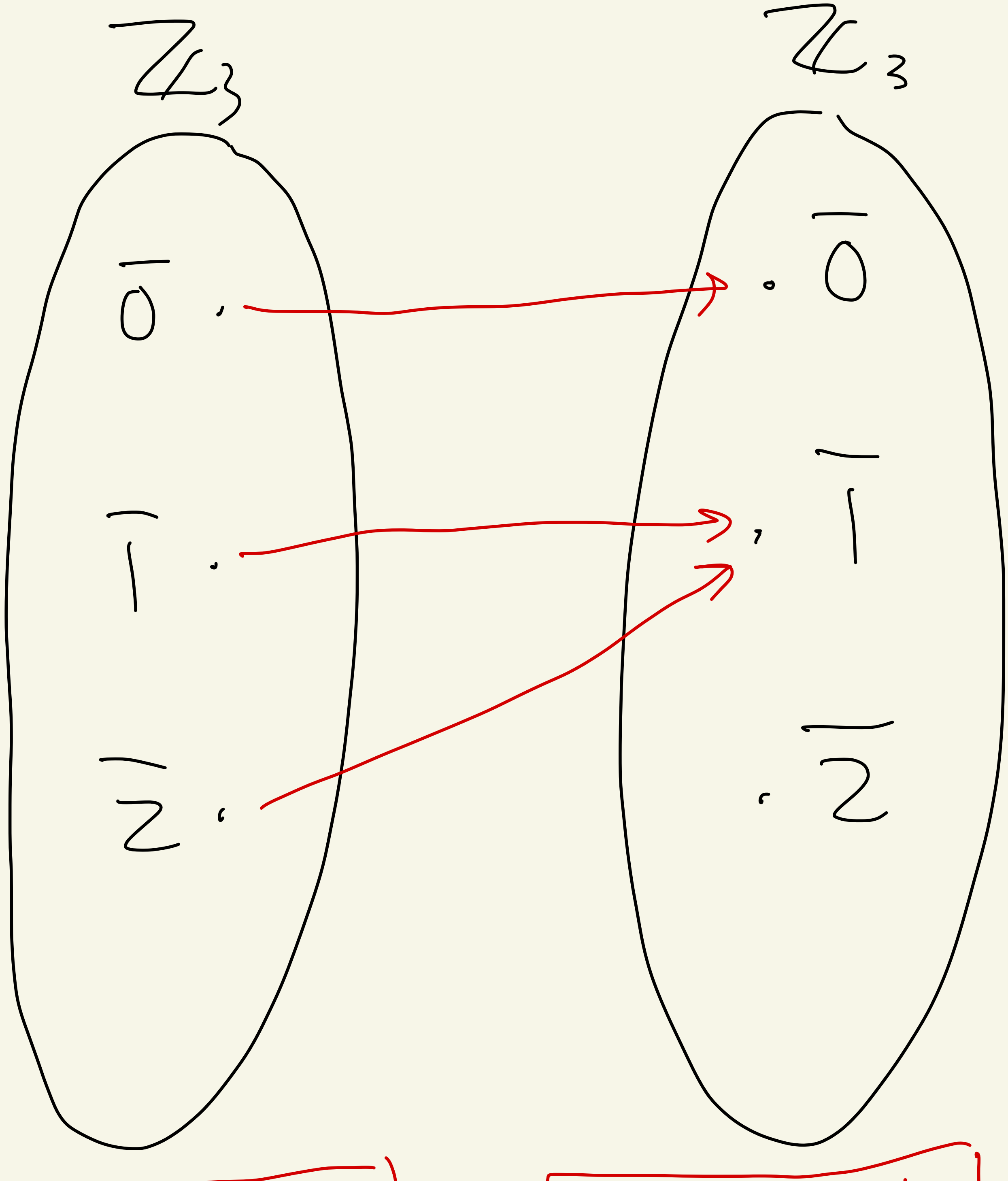
claim

$\boxed{\text{Ex:}}$ $n = 2$, $f(\bar{x}) = \bar{x}^2$



$\boxed{\text{f is 1-1}}$

<u>Ex:</u> $n=3$, $f(\bar{x}) = \bar{x}^2$



$\boxed{f(\bar{2}) = \bar{4} = \bar{1}}$

$\boxed{\bar{2} = \overline{-1} \ \& \ (\overline{-1})^2 = \bar{1}}$

f is not 1-1

Claim: Let $f: \mathbb{Z}_n \to \mathbb{Z}_n$ given by $f(\overline{x}) = \overline{x}^2$. If $n > 2$, then f is not one-to-one.

proof of claim:

Note first that since $n > 2$ we know that $\overline{1} \neq \overline{-1}$

Why? Suppose $\overline{1} = \overline{-1}$.

Then, $1 \equiv -1 \pmod{n}$.

Thus, $n \mid (1 - (-1))$

Ie, $n \mid 2$.

Thus, $n = \pm 1, \pm 2$.

Can't happen since $n > 2$

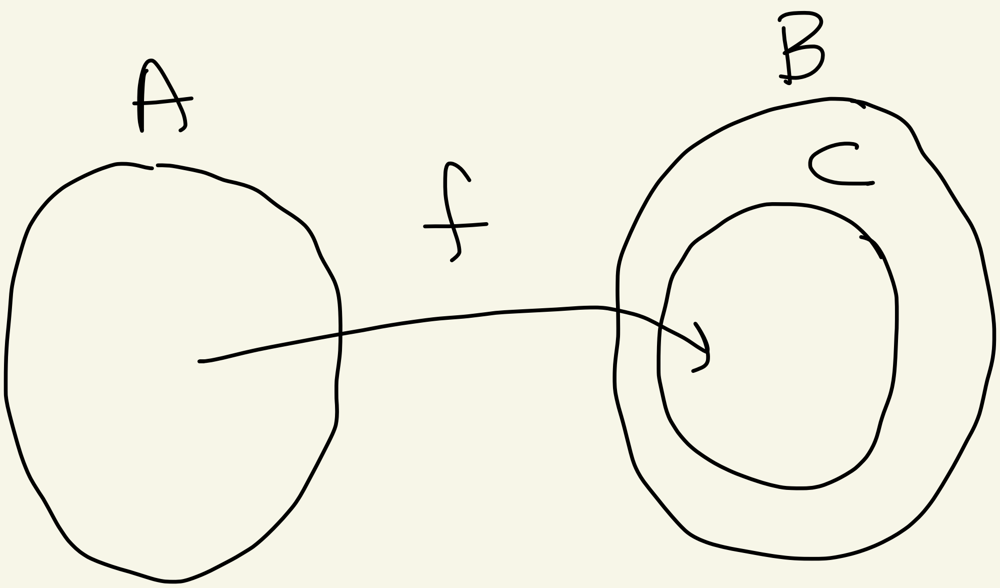
Thus, $\overline{1} \neq \overline{-1}$, however

$f(\overline{1}) = \overline{1}^2 = \overline{1}$ and

$f(\overline{-1}) = \overline{-1}^2 = \overline{1}$.

So, f is not 1-1 if $n > 2$. $\blacksquare$

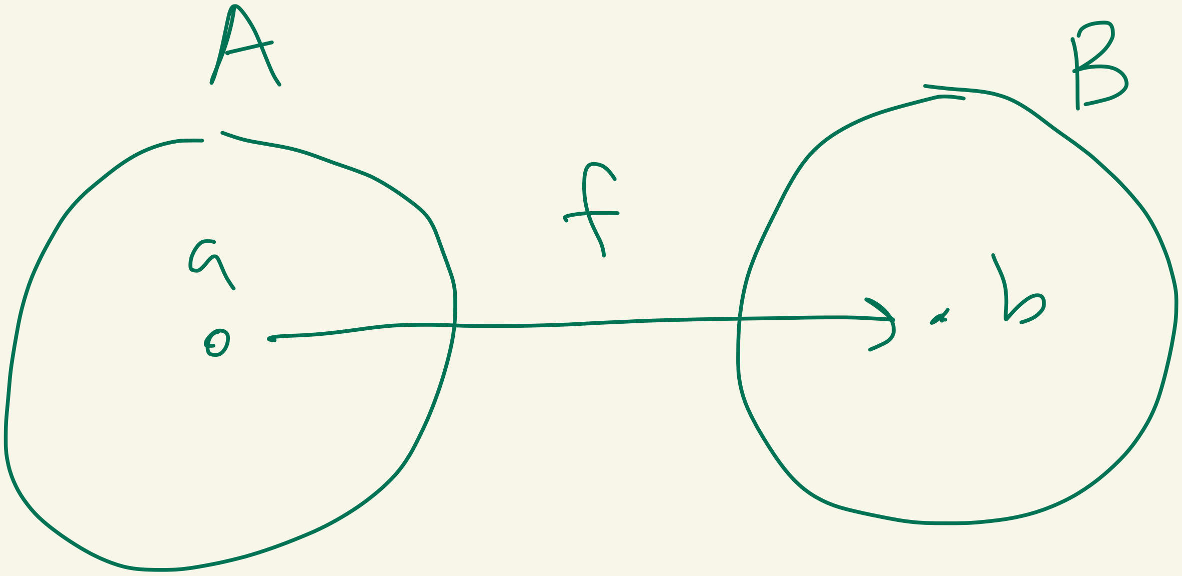
<u>Def:</u> Let A and B be sets. Let $f: A \to B$. Let C be the range of f. We say that f is <u>surjective</u> or <u>onto</u> B if $C = B$.



f is onto if C = B

<u>Another way to say:</u>

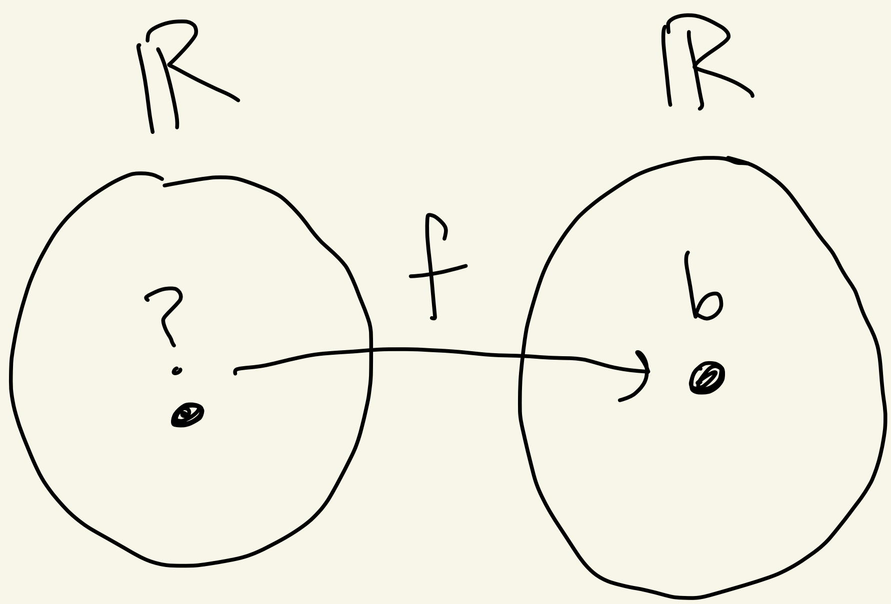
f is onto B if for every $b \in B$, there exists $a \in A$ with $f(a) = b$.



Ex: Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x) = -4x+5$.

Let's show that f is onto $\mathbb{R}$.

proof:



Let $b \in \mathbb{R}$.

We must

find $a \in \mathbb{R}$,
where $f(a) = b$.

Let $a = \frac{b-5}{-4}$.

Note $a \in \mathbb{R}$ and

$$f(a) = f\left(\frac{b-5}{-4}\right) = -4\left(\frac{b-5}{-4}\right) + 5$$

$$= (b-5) + 5 = b.$$

Thus, f is onto $\mathbb{R}$. $\blacksquare$

Scratchwork

$$f(a) = b$$

$$-4a + 5 = b$$

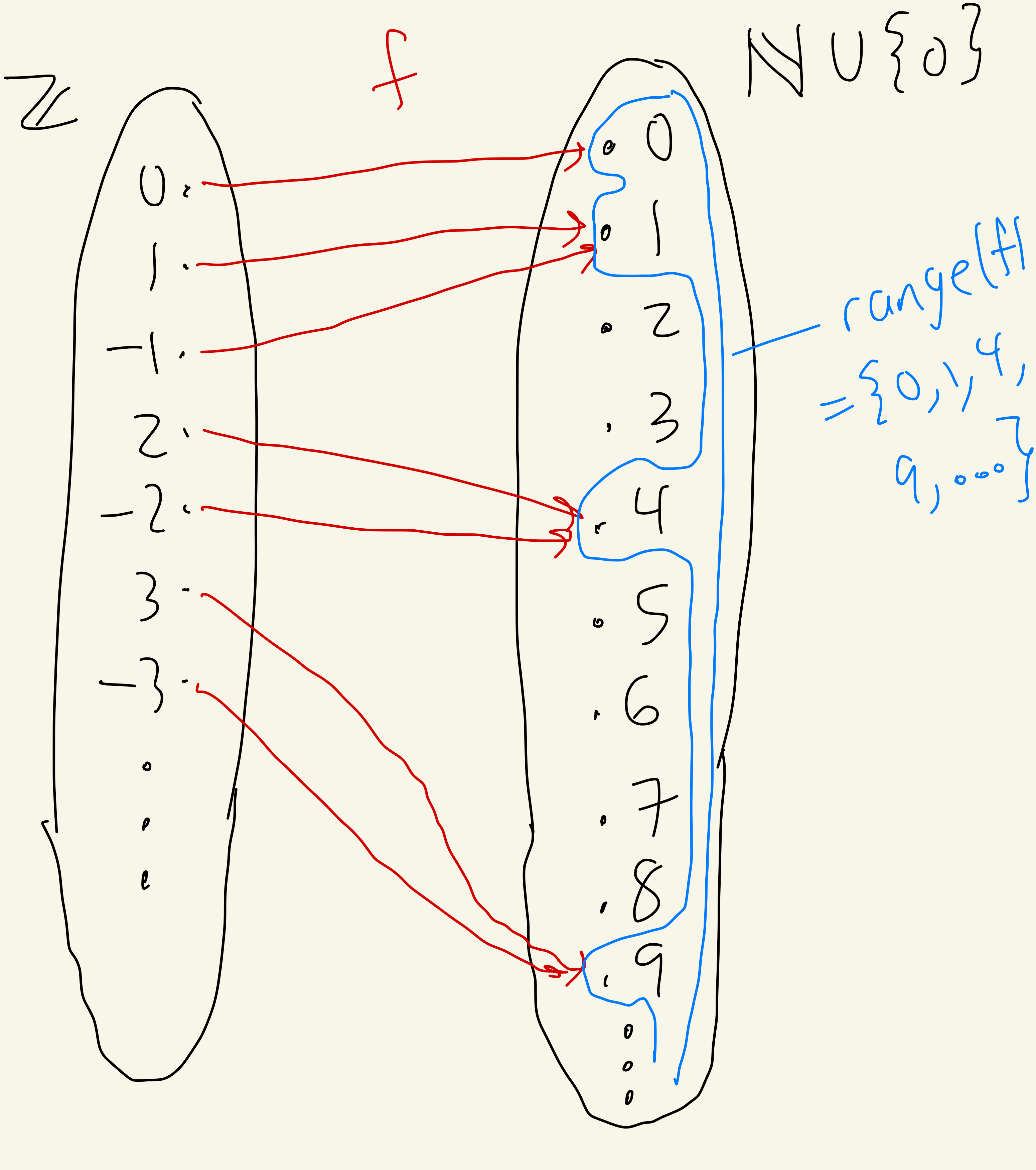
$$a = \frac{b-5}{-4}$$

How to show $f : A \to B$ is <u>not</u> onto

Find some $b \in B$ where is no $a \in A$ with $f(a) = b$

Ex: $f : \mathbb{Z} \to \mathbb{N} \cup \{0\}$

$f(x) = x^2$



f is not onto:

proof: Let $b = 2$.

Then, $b \in \mathbb{N} \cup \{0\}$.

But is no $a \in \mathbb{Z}$ with

$f(a) = 2$.

Why?

If so, then $a^2 = 2$.

Then, $a = \pm\sqrt{2} \notin \mathbb{Z}$.

Thus, f is not onto

because $2 \notin \text{range}(f)$.

$\blacksquare$