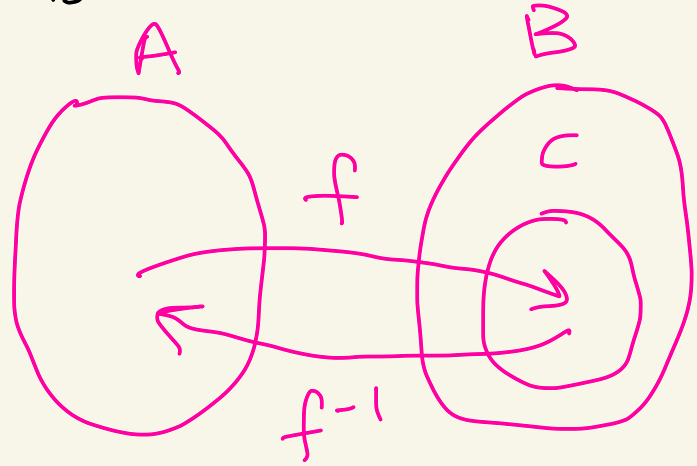
Math 3450
3/28/24

<u>Theorem:</u> Let A, B be sets. Let $f: A \to B$ be a one-to-one function. Let $C = \text{range}(f)$. Let $f^{-1}: C \to A$ be the inverse of f. Then:



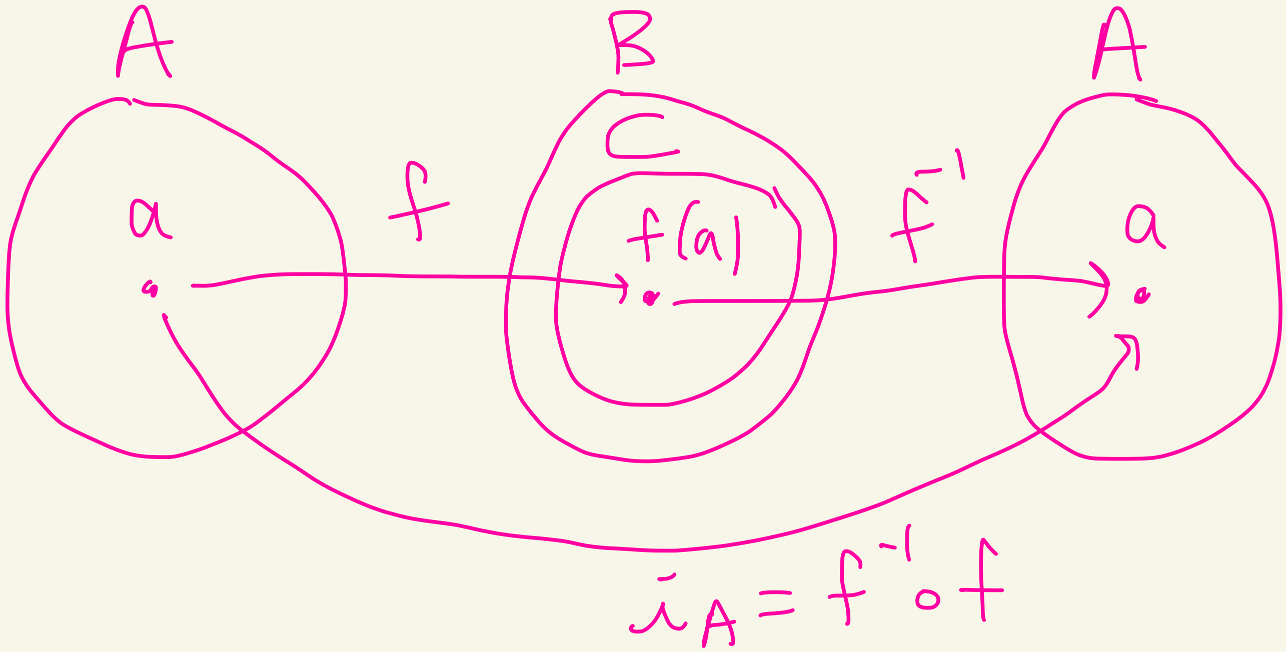
① $\text{domain}(f^{-1}) = \text{range}(f) = C$

② $\text{range}(f^{-1}) = \text{domain}(f) = A$
In particular, f^{-1} is onto A.

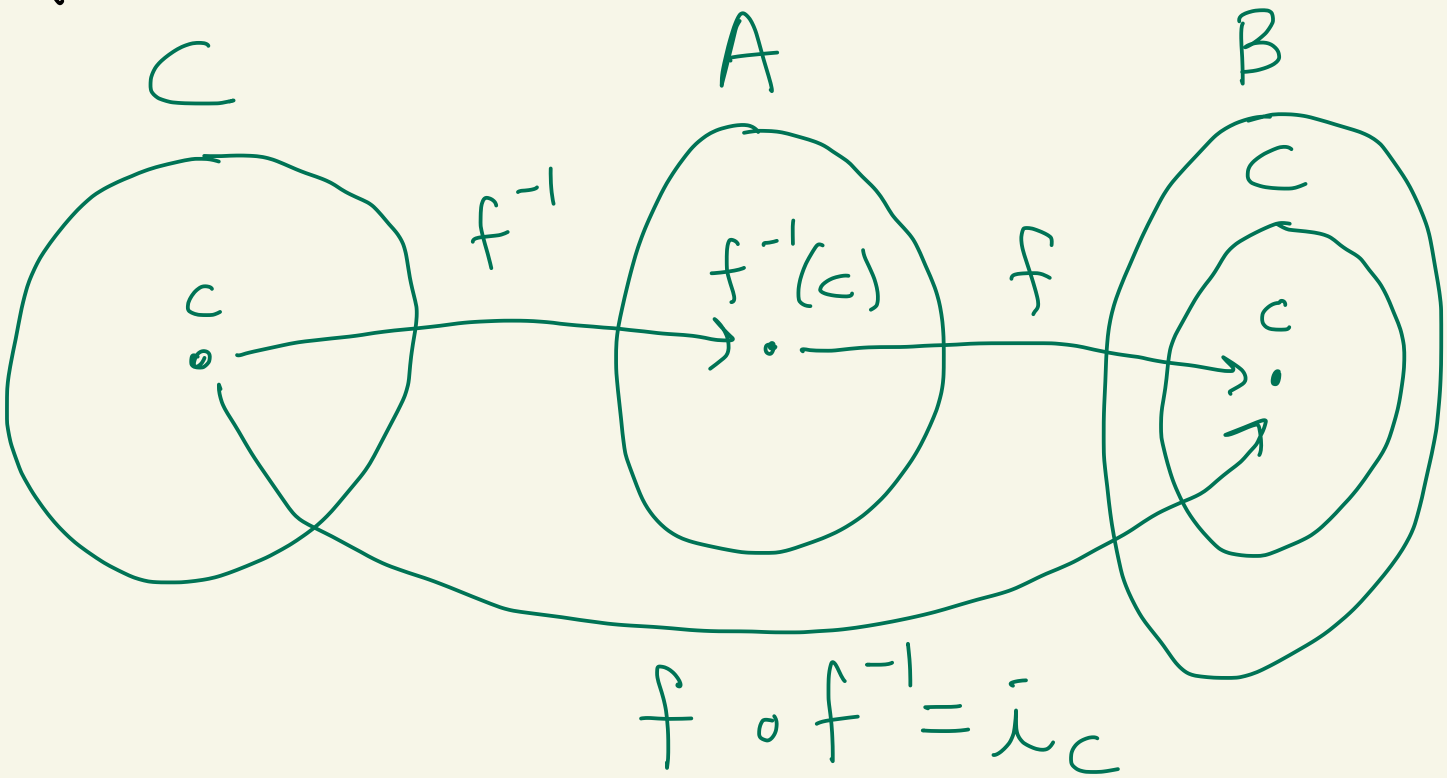
③ f^{-1} is one-to-one

④ $(f^{-1} \circ f)(a) = a$ for all $a \in A$.

So, $f^{-1} \circ f = i_A$



⑤ $(f \circ f^{-1})(c) = c$ for all $c \in C$.



⑥ If $g: C \to A$ and $g \circ f = i_A$, then $g = f^{-1}$.

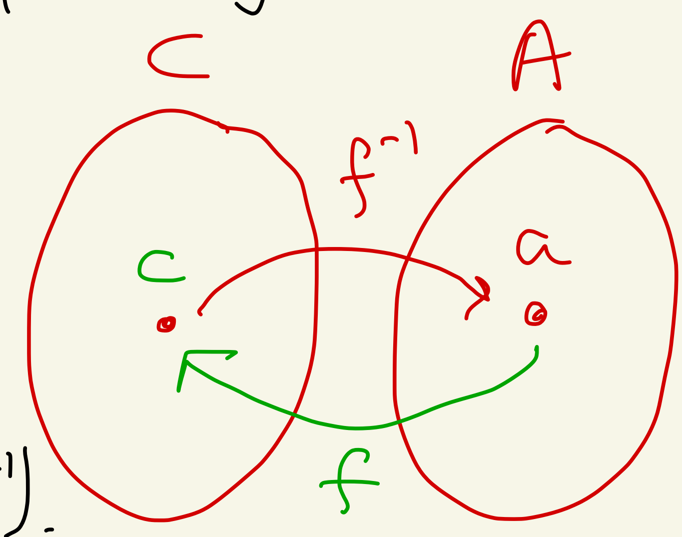
[⑥ Is a way to check that $g = f^{-1}$]

proof:

① By def of f^{-1} we have $\text{domain}(f^{-1}) = C = \text{range}(f)$.

② Let's show that $\text{range}(f^{-1}) = A$.

By def of f^{-1} we know $\text{range}(f^{-1}) \subseteq A$.



Why is $A \subseteq \text{range}(f^{-1})$.

Let $a \in A$.

Let $c = f(a)$

And, $f^{-1}(c) = a$ by def of f^{-1}.

So, $a \in \text{range}(f^{-1})$.

Thus, $A \subseteq \text{range}(f^{-1})$

Therefore, $A = \text{range}(f^{-1})$.

(3) Let's show that f^{-1} is one-to-one.

Suppose $f^{-1}(c_1) = f^{-1}(c_2)$ where $c_1, c_2 \in C$.

We need to show that $c_1 = c_2$.

Let $a = f^{-1}(c_1) = f^{-1}(c_2)$.

Since $a = f^{-1}(c_1)$ we know that $f(a) = c_1$.

Since $a = f^{-1}(c_2)$ we know that $f(a) = c_2$.

So, $c_1 = f(a) = c_2$.

Thus, f^{-1} is one-to-one.

(4) Let's show that $f^{-1} \circ f = i_A$.

Let $a \in A$.

Set $c = f(a)$.

So, $f^{-1}(c) = a$ by def of f^{-1}.

Then,

$$\begin{aligned}(f^{-1} \circ f)(a) &= f^{-1}(f(a)) \\ &= f^{-1}(c) \\ &= a \\ &= i_A(a)\end{aligned}$$

Thus, $(f^{-1} \circ f)(a) = i_A(a)$ for all $a \in A$.

So, $f^{-1} \circ f = i_A$

(5) Let's show that
$(f \circ f^{-1})(c) = c$ for all $c \in C$.

Let $c \in C$.
Then, $f^{-1}(c) = a$ where $a \in A$
and $f(a) = c$.

Thus,

$$\begin{aligned}(f \circ f^{-1})(c) &= f(f^{-1}(c)) \\ &= f(a) \\ &= c \\ &= i_C(c)\end{aligned}$$

⑥ Let $g: C \to A$ where $g \circ f = i_A$

We want to show that $g = f^{-1}$.

So we must show that

$g(c) = f^{-1}(c)$ for all $c \in C$.

Let $c \in C$.

Then, $f^{-1}(c) = a$ where

$a \in A$ and $f(a) = c$.

Then,

$$g(c) = g(f(a)) = (g \circ f)(a)$$

assumption $g \circ f = i_A$

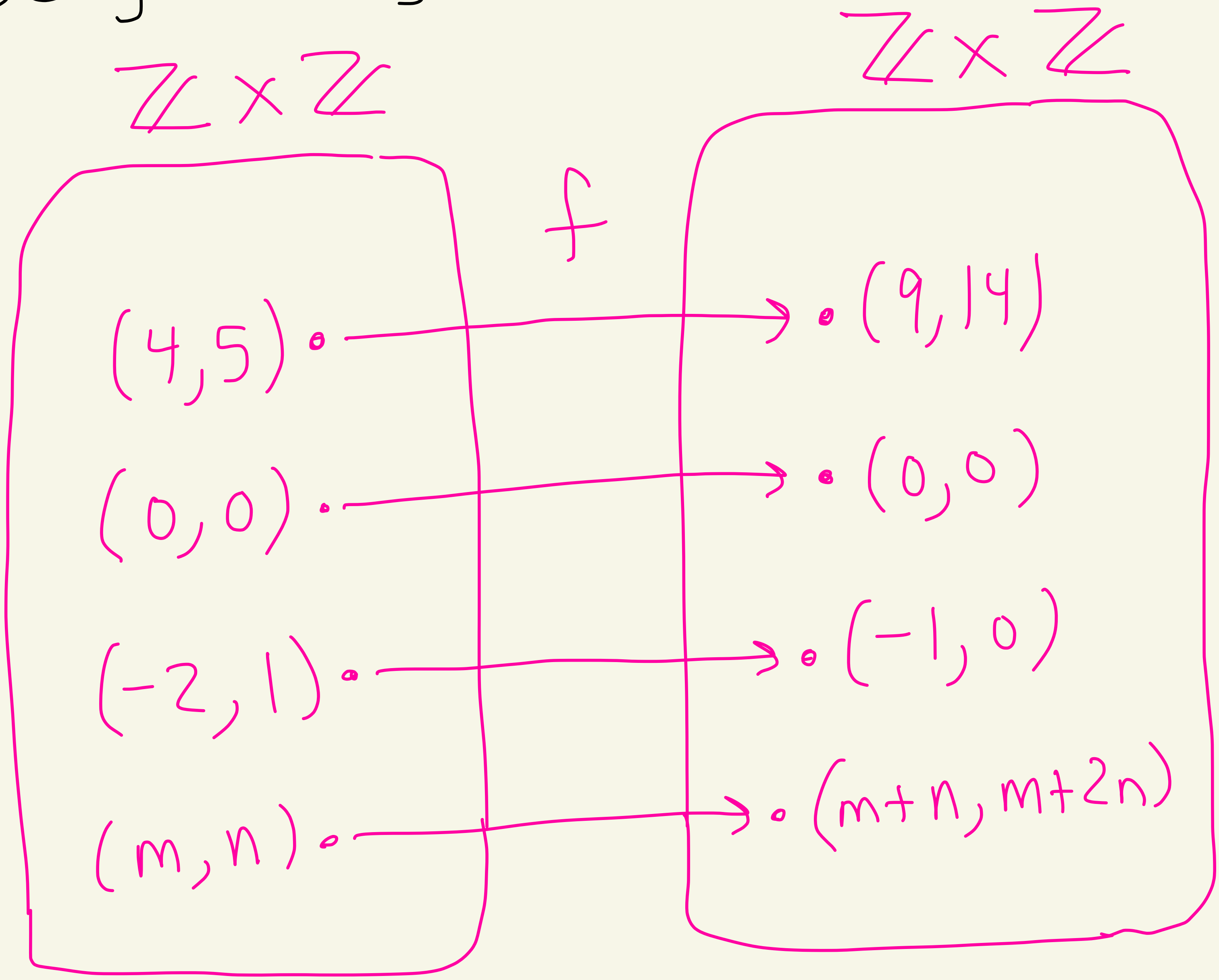
$$= i_A(a)$$

$$= a$$

$$= f^{-1}(c)$$

Thus, $g = f^{-1}$. $\blacksquare$

<u>Ex:</u> Let $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be given by $f(m,n) = (m+n, m+2n)$



$$f(4,5) = (4+5, 4+2\cdot 5) = (9,14)$$

$$f(-2,1) = (-2+1, -2+2\cdot 1) = (-1,0)$$

$$\boxed{\text{Claim: } f \text{ is one-to-one}}$$

<u>proof:</u>

Suppose $f(m_1, n_1) = f(m_2, n_2)$ where $(m_1, n_1), (m_2, n_2) \in \mathbb{Z} \times \mathbb{Z}$.

We need to show that $(m_1, n_1) = (m_2, n_2)$.

Since $f(m_1, n_1) = f(m_2, n_2)$ we know that $(m_1 + n_1, m_1 + 2n_1) = (m_2 + n_2, m_2 + 2n_2)$.

Thus,

$$\boxed{\begin{aligned} m_1 + n_1 &= m_2 + n_2 \quad (1) \\ m_1 + 2n_1 &= m_2 + 2n_2 \quad (2) \end{aligned}}$$

Calculating 2 − 1 we get that $n_1 = n_2$.

Thus we get

$$m_1 + n_2 \underset{\boxed{n_2 = n_1}}{=} m_1 + n_1 \underset{\boxed{\text{eqn } 1}}{=} m_2 + n_2$$

Subtract n_2 from both sides to get $m_1 = m_2$.

Thus, $(m_1, n_1) = (m_2, n_2)$.

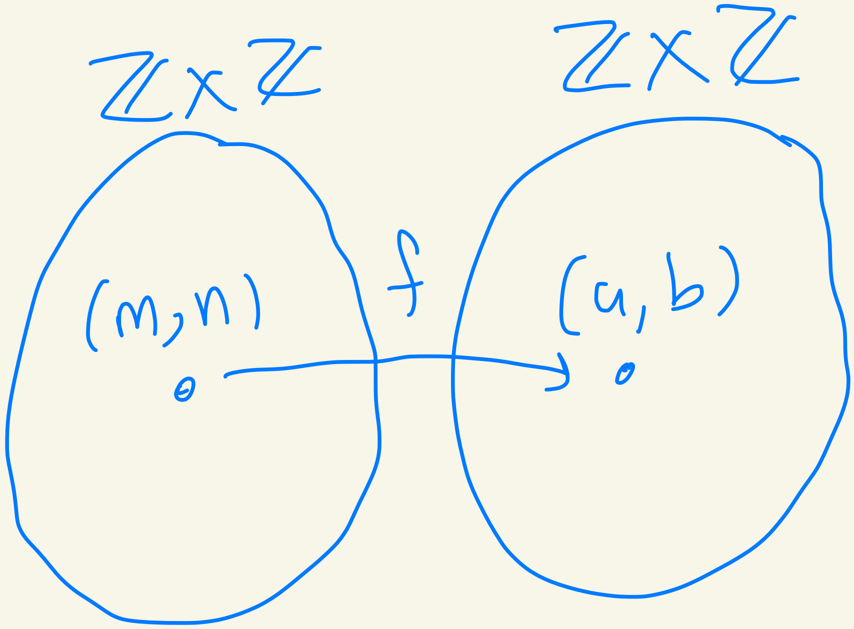
Thus, f is one-to-one.

$\boxed{\text{claim } 1}$

$\underline{\text{Claim 2:}}$ f is onto

Let $(a,b) \in \mathbb{Z} \times \mathbb{Z}$.

We must find $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ where $f(m,n) = (a,b)$



That is, we need to solve

$$\underbrace{(m+n,\ m+2n)}_{f(m,n)} = (a,b).$$

So we need to solve

$$\boxed{\begin{aligned} m+n &= a \\ m+2n &= b \end{aligned}} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

for m and n.

Calculating (2) $-$ (1) you get

that $n = b-a$.

Then,

$$m \underset{\text{eqn (1)}}{=} a - n \underset{n=b-a}{=} a-(b-a) = 2a-b.$$

So, set $(m,n) = \underbrace{(2a-b,\ b-a)}_{\text{this is in } \mathbb{Z}\times\mathbb{Z} \text{ because } a,b\in\mathbb{Z}.}$.

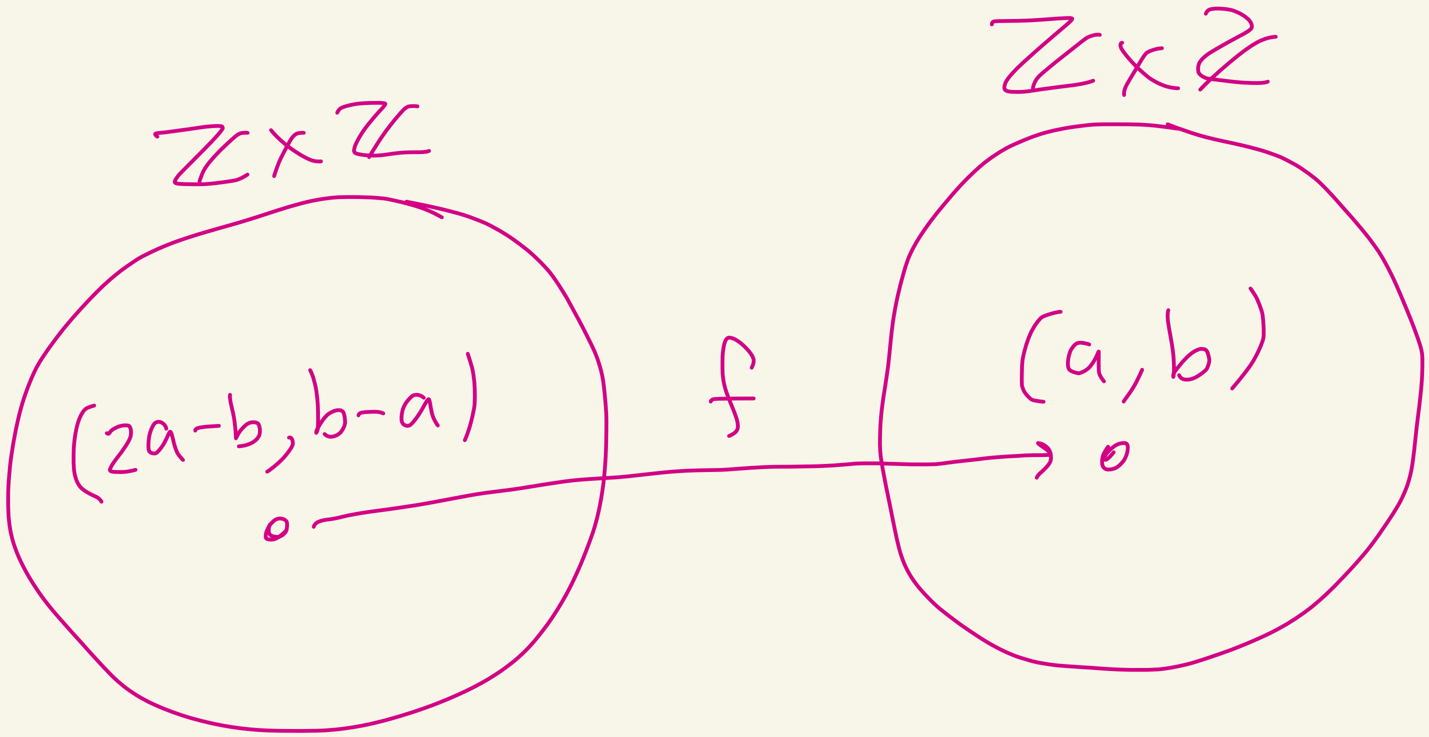
And we have that

$$f(m,n) = f(2a-b,\ b-a)$$

$$= (2a-b+b-a,\ 2a-b+2(b-a))$$

$$= (a, b)$$

Thus, f is onto.



$\boxed{\text{Claim 2}}$ —