Math 3450

3/7/24

Well-defined functions

Ex: Suppose you and your friend Francis want to define a function on $\mathbb{Q}$. You say "How about this function? $f: \mathbb{Q} \to \mathbb{Q}$ where

$$f\left(\frac{a}{b}\right) = \frac{b}{a}$$ "

Francis says "I don't know about that function. What about $f\left(\frac{0}{1}\right) = \frac{1}{0}$? That doesn't seem to make sense."

You say "You're right. Good call."

Then you say, "Ok I've got

another idea. How about $g: \mathbb{Q} \to \mathbb{Q}$ where $g\left(\frac{a}{b}\right) = a$?

That totally works. For example, $g\left(\frac{3}{5}\right) = 3$ and $g\left(\frac{0}{2}\right) = 0$."

Then Francis says, "Hey wait a minute, $g\left(\frac{3}{5}\right) = 3$ but $g\left(\frac{6}{10}\right) = 6$ and $\frac{3}{5} = \frac{6}{10}$. Shouldn't g agree on those numbers?"

You say "oh yeah you're right."

The functions f and g above are not well-defined.

How to check that $f: A \to B$ is well-defined

Check two things:

1. If $a \in A$, then $f(a) \in B$
2. If some or all of the elements from A can be expressed in more than one way then we must check that if a_1, a_2 are two expressions of the same element in A (ie $a_1 = a_2$) then $f(a_1) = f(a_2)$

<u>Ex:</u> Let $f : \mathbb{Q} \to \mathbb{Q}$ where
$f\left(\frac{a}{b}\right) = \left(\frac{a}{b}\right)^2$.

Is f well-defined? Yes!

<u>proof that f is well-defined:</u>

① Let $\frac{a}{b} \in \mathbb{Q}$.

So, $a, b \in \mathbb{Z}$ and $b \neq 0$.

Then, $f\left(\frac{a}{b}\right) = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$

We have that $a^2, b^2 \in \mathbb{Z}$
and $b^2 \neq 0$ (since $b \neq 0$).

So, $\frac{a^2}{b^2} \in \mathbb{Q}$.

② Suppose $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ and $\frac{a}{b} = \frac{c}{d}$.

Is $f\left(\frac{a}{b}\right) = f\left(\frac{c}{d}\right)$?

Method 1:

Since $\frac{a}{b} = \frac{c}{d}$, then by squaring both sides we get $\underbrace{\left(\frac{a}{b}\right)^2 = \left(\frac{c}{d}\right)^2}$.

So, $f\left(\frac{a}{b}\right) = f\left(\frac{c}{d}\right)$

You might ask, why is this true?

Method 2:

Recall how we define two fractions to be equal:

$$\frac{w}{x} = \frac{y}{z} \quad \text{means} \quad wz = xy$$

Suppose $\frac{a}{b} = \frac{c}{d}$.

Then $ad = bc$.

So, $(ad)^2 = (bc)^2$ — using integer mult. is well-defined

Then, $a^2d^2 = b^2c^2$

So, $\frac{a^2}{b^2} = \frac{c^2}{d^2}$

Thus, $f\left(\frac{a}{b}\right) = f\left(\frac{c}{d}\right)$

From 1 and 2 above f is well-defined. $\blacksquare$

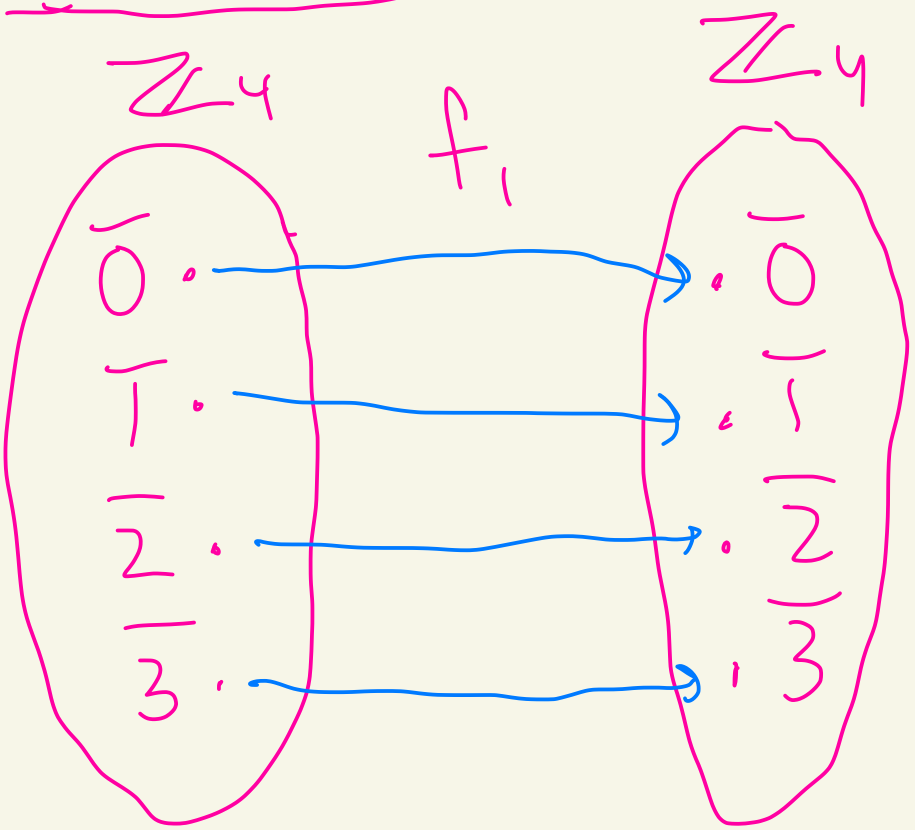
<u>Ex:</u> Let $n \in \mathbb{Z}$, $n \geq 2$.

Pick $a \in \mathbb{Z}$.

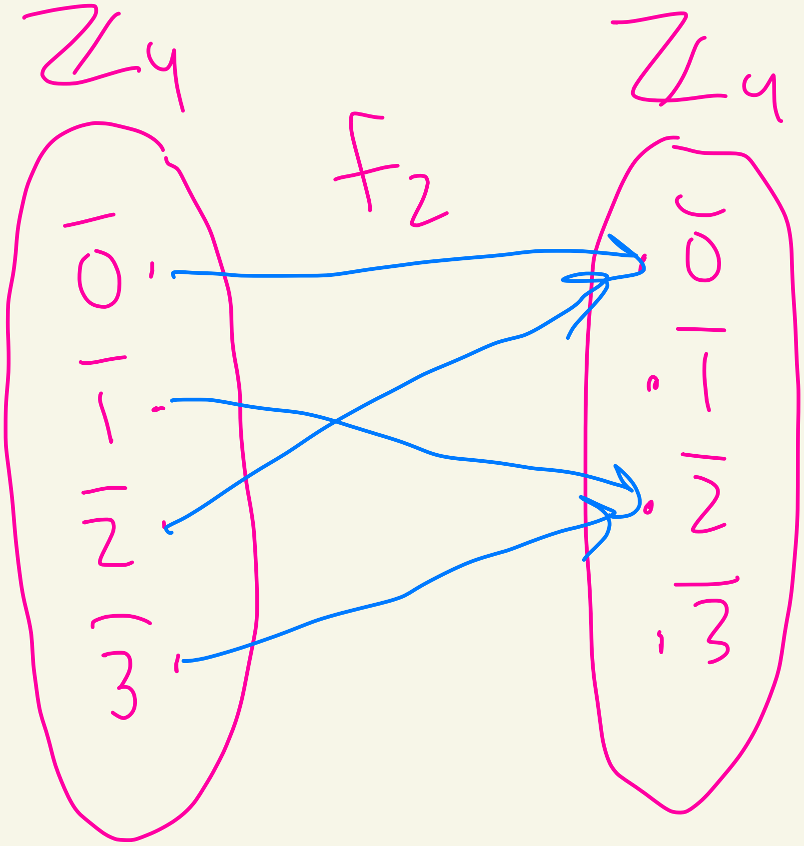
Define $f_a : \mathbb{Z}_n \to \mathbb{Z}_n$

by $f_a(\overline{x}) = \overline{a} \cdot \overline{x}$

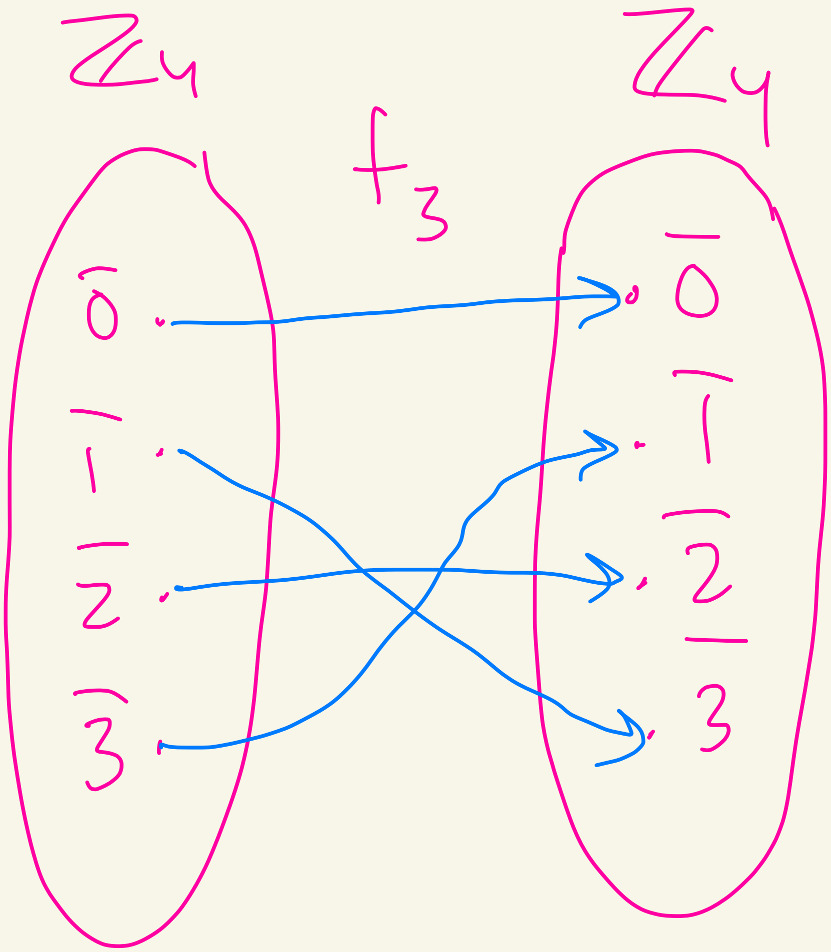
Let's do some examples when $n=4$, $\mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$



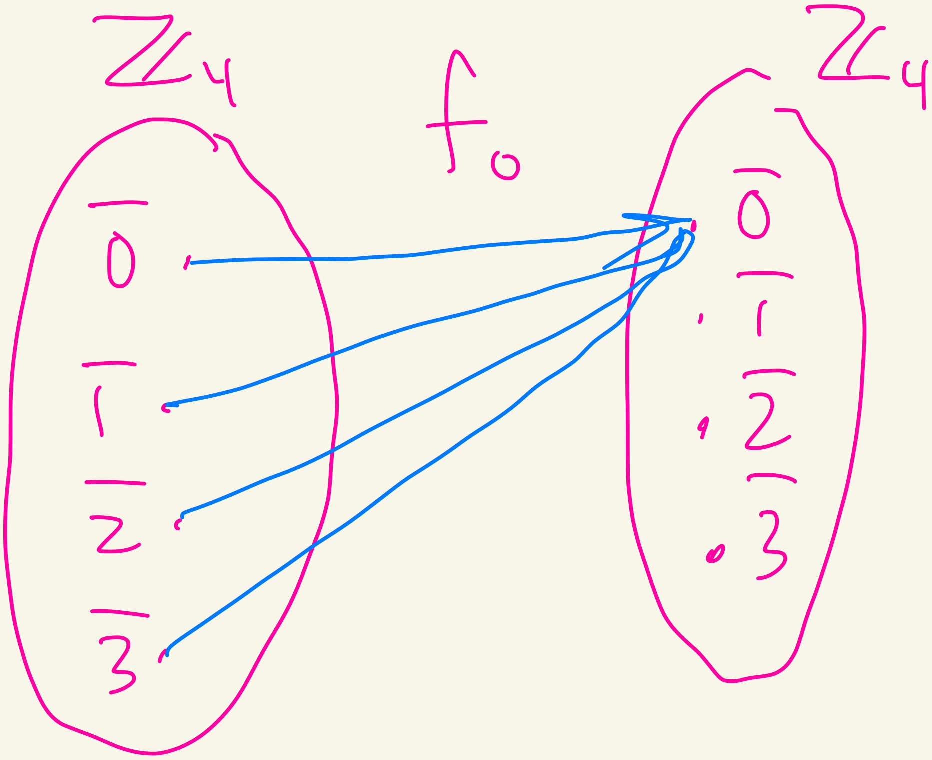
$$\boxed{\begin{aligned} f_1(\overline{0}) &= \overline{1} \cdot \overline{0} = \overline{0} \\ f_1(\overline{1}) &= \overline{1} \cdot \overline{1} = \overline{1} \\ f_1(\overline{2}) &= \overline{1} \cdot \overline{2} = \overline{2} \\ f_1(\overline{3}) &= \overline{1} \cdot \overline{3} = \overline{3} \end{aligned}}$$



$$f_2(\bar{0}) = \bar{2}\cdot\bar{0} = \bar{0}$$
$$f_2(\bar{1}) = \bar{2}\cdot\bar{1} = \bar{2}$$
$$f_2(\bar{2}) = \bar{2}\cdot\bar{2} = \bar{4} = \bar{0}$$
$$f_2(\bar{3}) = \bar{2}\cdot\bar{3} = \bar{6} = \bar{2}$$



$$f_3(\bar{0}) = \bar{3}\cdot\bar{0} = \bar{0}$$
$$f_3(\bar{1}) = \bar{3}\cdot\bar{1} = \bar{3}$$
$$f_3(\bar{2}) = \bar{3}\cdot\bar{2} = \bar{6} = \bar{2}$$
$$f_3(\bar{3}) = \bar{3}\cdot\bar{3} = \bar{9} = \bar{1}$$



$$\boxed{\begin{aligned} f_0(\bar{x}) &= \bar{0}\cdot\bar{x} \\ &= \bar{0} \end{aligned}}$$

Theorem: Let $n \in \mathbb{Z}$, $n \geq 2$. Let $a \in \mathbb{Z}$. Let $f_a : \mathbb{Z}_n \to \mathbb{Z}_n$ be given by $f_a(\bar{x}) = \bar{a}\cdot\bar{x}$. Then f_a is well-defined.

proof:

① Let $\bar{x} \in \mathbb{Z}_n$ where $x \in \mathbb{Z}$.

Since $x, a \in \mathbb{Z}$ we know $ax \in \mathbb{Z}$.

Thus,

$$f_a(\overline{x}) = \overline{a} \cdot \overline{x} = \overline{ax} \in \mathbb{Z}_n.$$

(2) Let $\overline{x}, \overline{y} \in \mathbb{Z}_n$ where $\overline{x} = \overline{y}$.

Then,

since $\overline{x} = \overline{y}$

$$f_a(\overline{x}) = \overline{a} \cdot \overline{x} = \overline{a} \cdot \overline{y} = f_a(\overline{y}).$$

when we talked about well-defined operations we proved that if $\overline{b} = \overline{c}$ and $\overline{d} = \overline{e}$, then $\overline{b} \cdot \overline{d} = \overline{c} \cdot \overline{e}$

$\blacksquare$

Test review

Hammack, Ch 8

(18) Prove $A \times (B - C) = (A \times B) - (A \times C)$ where A, B, C are sets.

Proof:

$(\subseteq)$: Let $w \in A \times (B - C)$

Then, $w = (x, y)$ where $x \in A$ and $y \in B - C$.

So, $x \in A$ and $y \in B$ and $y \notin C$.

Then, $(x, y) \in A \times B$ ← since $x \in A$ $y \in B$

and $(x,y) \notin A \times C$ ← since $y \notin C$

Thus, $(x,y) \in (A \times B) - (A \times C)$

Therefore,

$$A \times (B - C) \subseteq (A \times B) - (A \times C).$$

$\boxed{2}$: Let $z \in (A \times B) - (A \times C)$.

Then, $z \in (A \times B)$ and

$z \notin (A \times C)$.

So, $z = (x,y)$ where

$x \in A$ and $y \in B$

and $y \notin C$.

$\boxed{\text{techniquely } z = (x,y) \notin A \times C}$

means $x \notin A$ or $y \notin C$
but we know $x \in A$
since $z \in A \times B$ so
we can conclude that $y \notin C$

Thus,

$$z = (x, y) \in A \times (B - C)$$

since
$x \in A$
$y \in B$
$y \notin C$

Hence

$$(A \times B) - (A \times C) \subseteq A \times (B - C).$$

$\blacksquare$