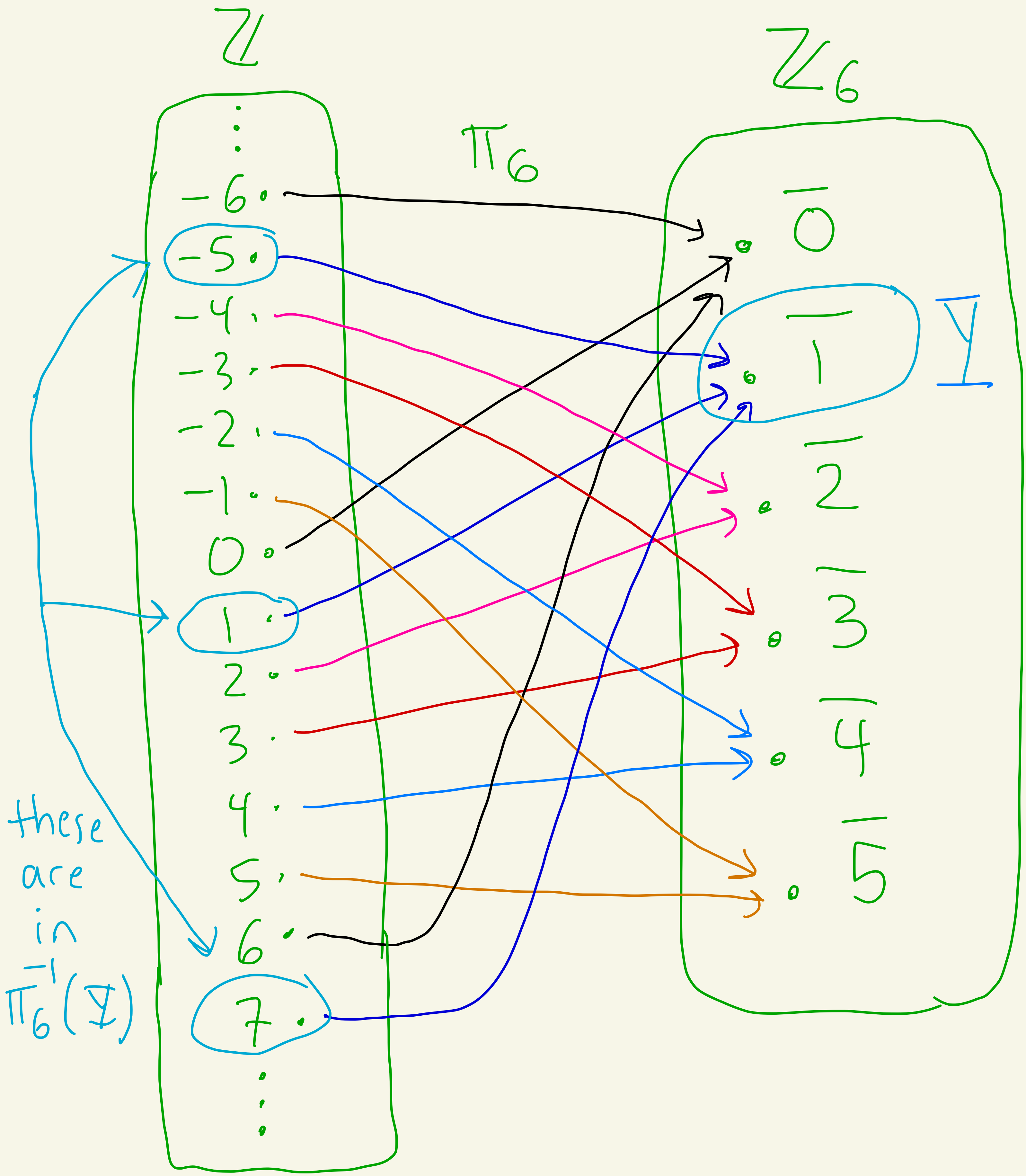
Math 3450
4/11/24

Let's calculate $\pi_6^{-1}(\overline{Y})$ where $\overline{Y} = \{\overline{1}\}$



Note: $-5, 1, 7 \in \pi_6^{-1}(\overline{1})$

And,

$$-5 = 6(-1) + 1$$
$$1 = 6(0) + 1$$
$$7 = 6(1) + 1$$

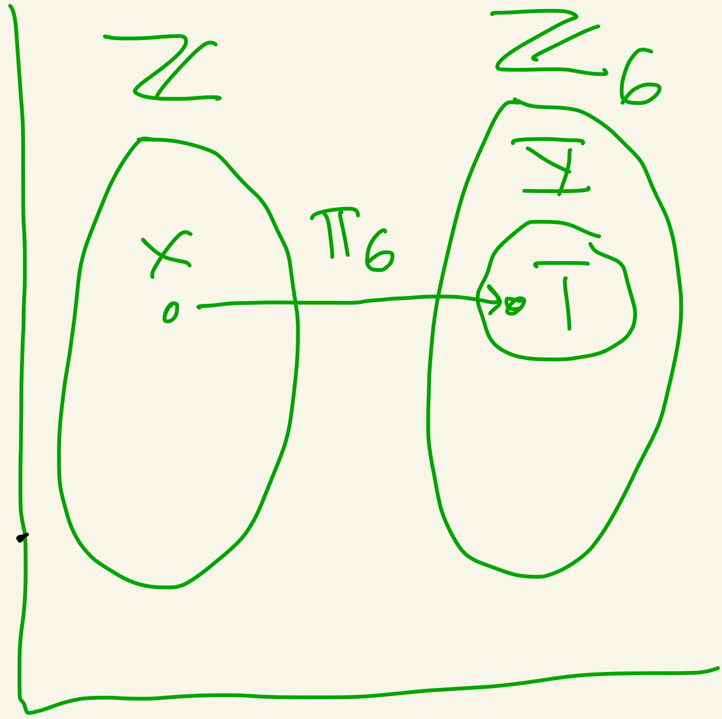
Also, $13 \in \pi_6^{-1}(\overline{1})$ and

$$13 = 6(2) + 1.$$

Claim: $\pi_6^{-1}(\overline{1}) = \{6k+1 \mid k \in \mathbb{Z}\}$

proof:

$(\subseteq)$: Let $x \in \pi_6^{-1}(\overline{1})$.



So, $\pi_6(x) \in Y$

Thus, $\pi_6(x) = \bar{1}$.

So, $\bar{x} = \bar{1}$ in $\mathbb{Z}_6$.

Then, $x \equiv 1 \pmod{6}$.

Thus, $6 \mid (x-1)$.

Hence, $x-1 = 6\ell$ where $\ell \in \mathbb{Z}$.

Therefore, $x = 6\ell + 1$.

Thus, $x \in \{6k+1 \mid k \in \mathbb{Z}\}$

Hence, $\pi_6^{-1}(Y) \subseteq \{6k+1 \mid k \in \mathbb{Z}\}$

$(\supseteq)$: Let $y \in \{6k+1 \mid k \in \mathbb{Z}\}$

So, $y = 6l + 1$ where $l \in \mathbb{Z}$.

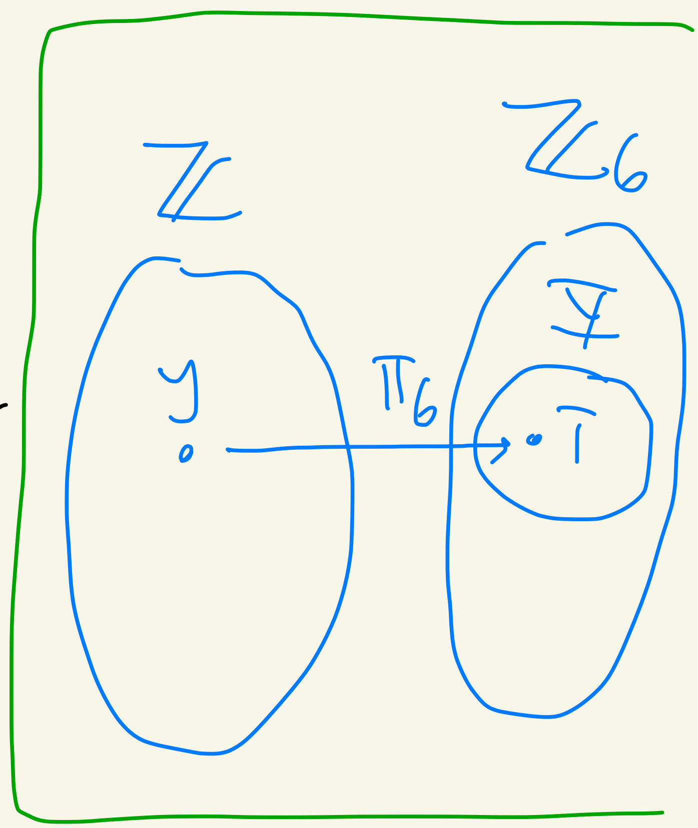
Then,

$$\pi_6(y) = \bar{y} = \overline{6l+1}$$
$$= \bar{6}\,\bar{l} + \bar{1}$$
$$= \bar{0}\,l + \bar{1}$$
$$= \bar{1}$$

$\bar{6} = \bar{0}$ in $\mathbb{Z}_6$

So, $\pi_6(y) \in Y$.

Thus, $y \in \pi_6^{-1}(Y)$.

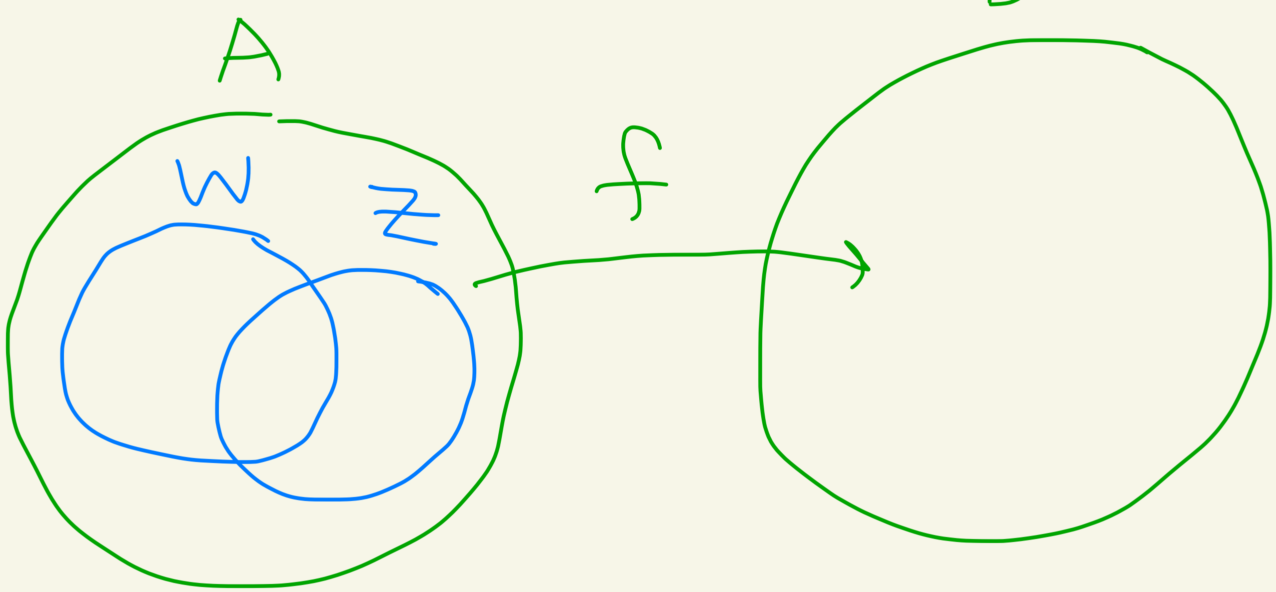


Therefore,

$\{6k+1 \mid k \in \mathbb{Z}\} \subseteq \pi_6^{-1}(Y)$.

By $(\subseteq)$ and $(\supseteq)$, $\pi_6^{-1}(Y) = \{6k+1 \mid k \in \mathbb{Z}\}$ $\blacksquare$

Theorem: Let A, B, W, Z be sets where $W \subseteq A$ and $Z \subseteq A$. Let $f: A \to B$.



Then:

① $f(W \cup Z) = f(W) \cup f(Z)$ — HW #4

② $f(W \cap Z) \subseteq f(W) \cap f(Z)$

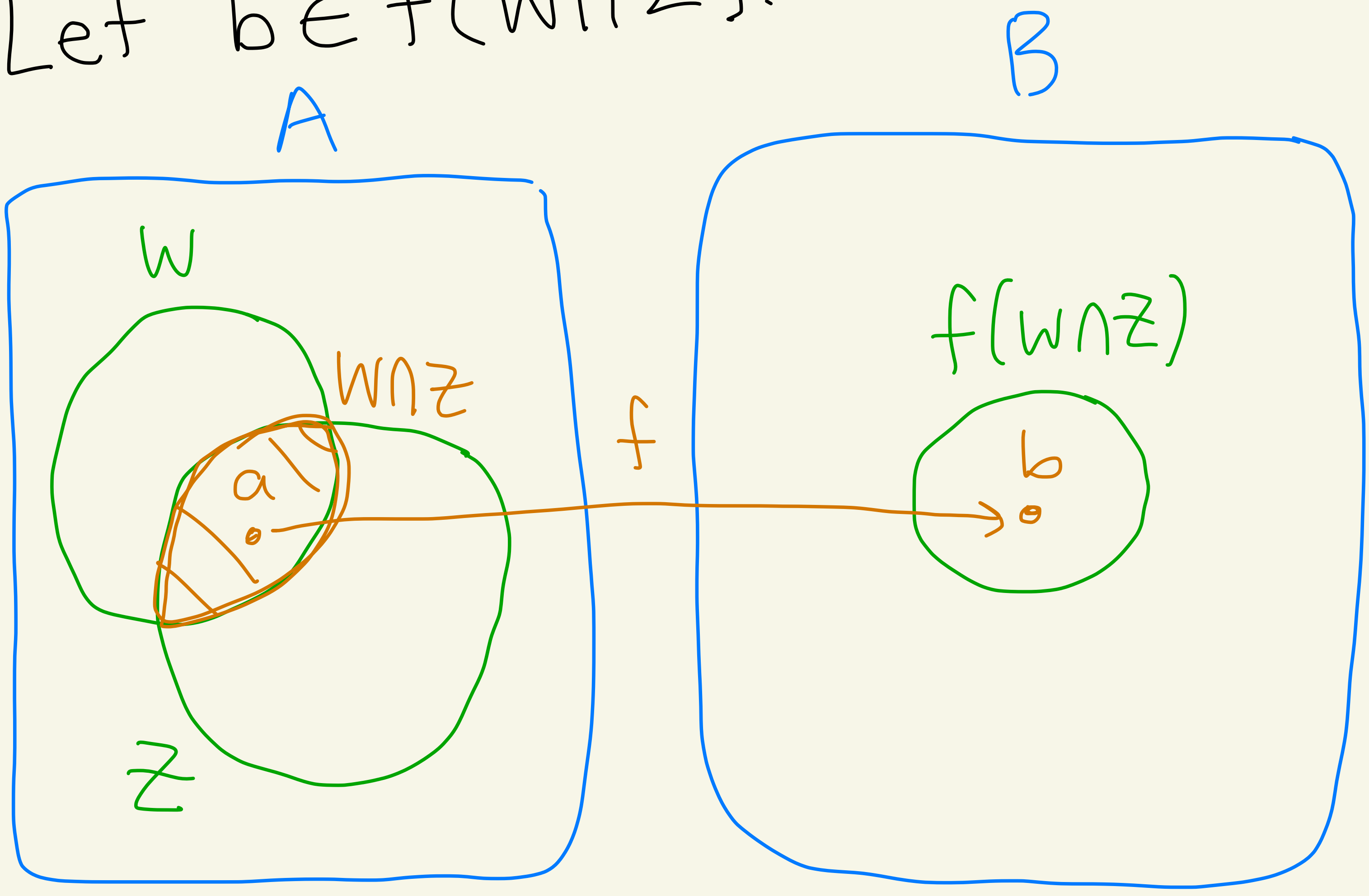
③ Give an example to show that $f(W \cap Z) = f(W) \cap f(Z)$ is not always true

Hammock 12.6 #7, 8

proof: Let's prove ②, ③, then ①

② We want to show that
$f(W \cap Z) \subseteq f(W) \cap f(Z)$.

Let $b \in f(W \cap Z)$.



Then there exists $a \in W \cap Z$
where $f(a) = b$.

Since $a \in W \cap Z$ we know
$a \in W$ and $a \in Z$.

Since $a \in W$ and $f(a) = b$
we know $b \in f(W)$

Since $a \in Z$ and $f(a) = b$
we know $b \in f(Z)$.

Thus, $b \in f(W) \cap f(Z)$.

Hence, $f(W \cap Z) \subseteq f(W) \cap f(Z)$.

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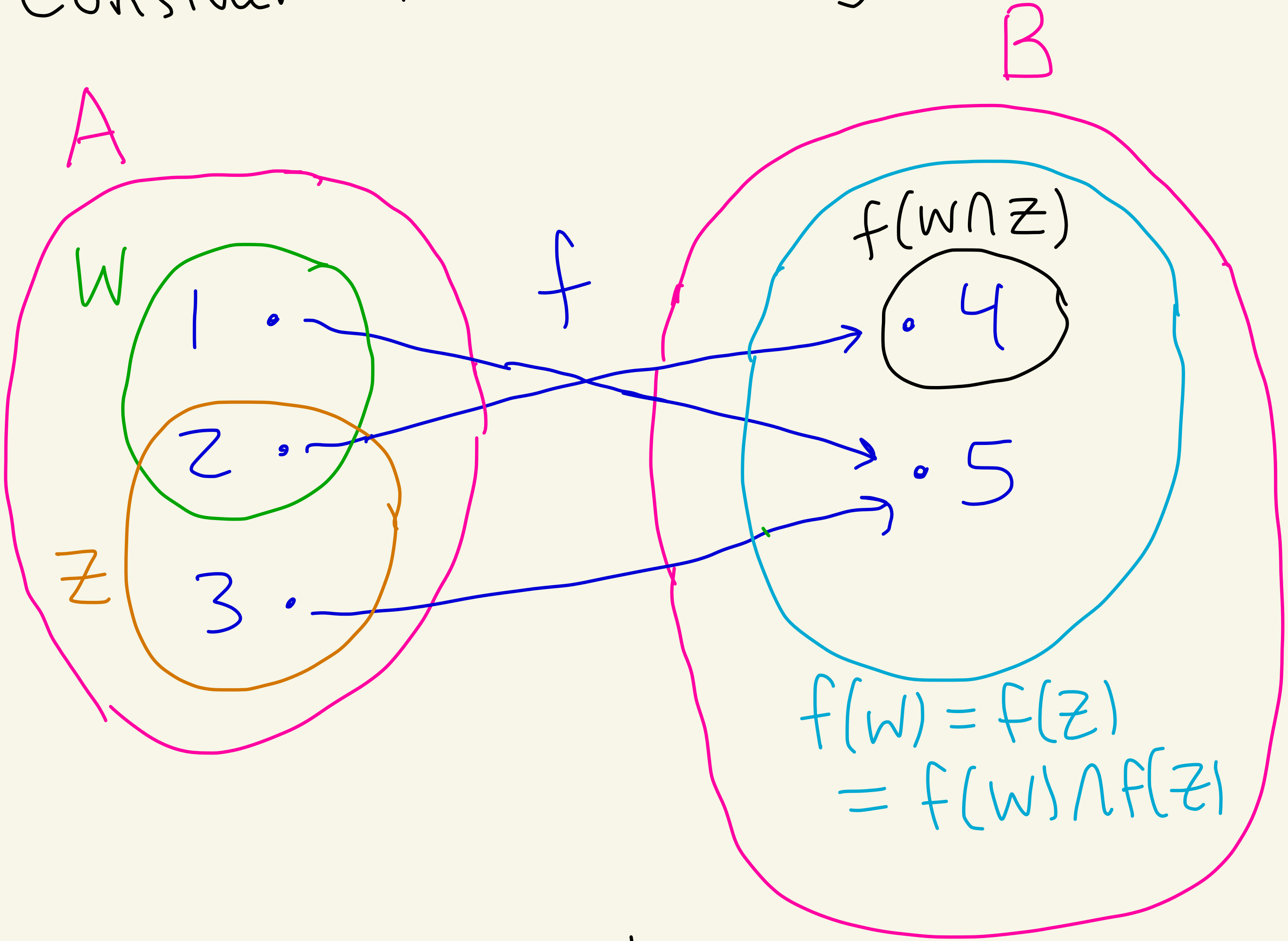
3 Let's give an example
to show that
$f(W \cap Z) = f(W) \wedge f(Z)$
is not always true.

Consider the following:



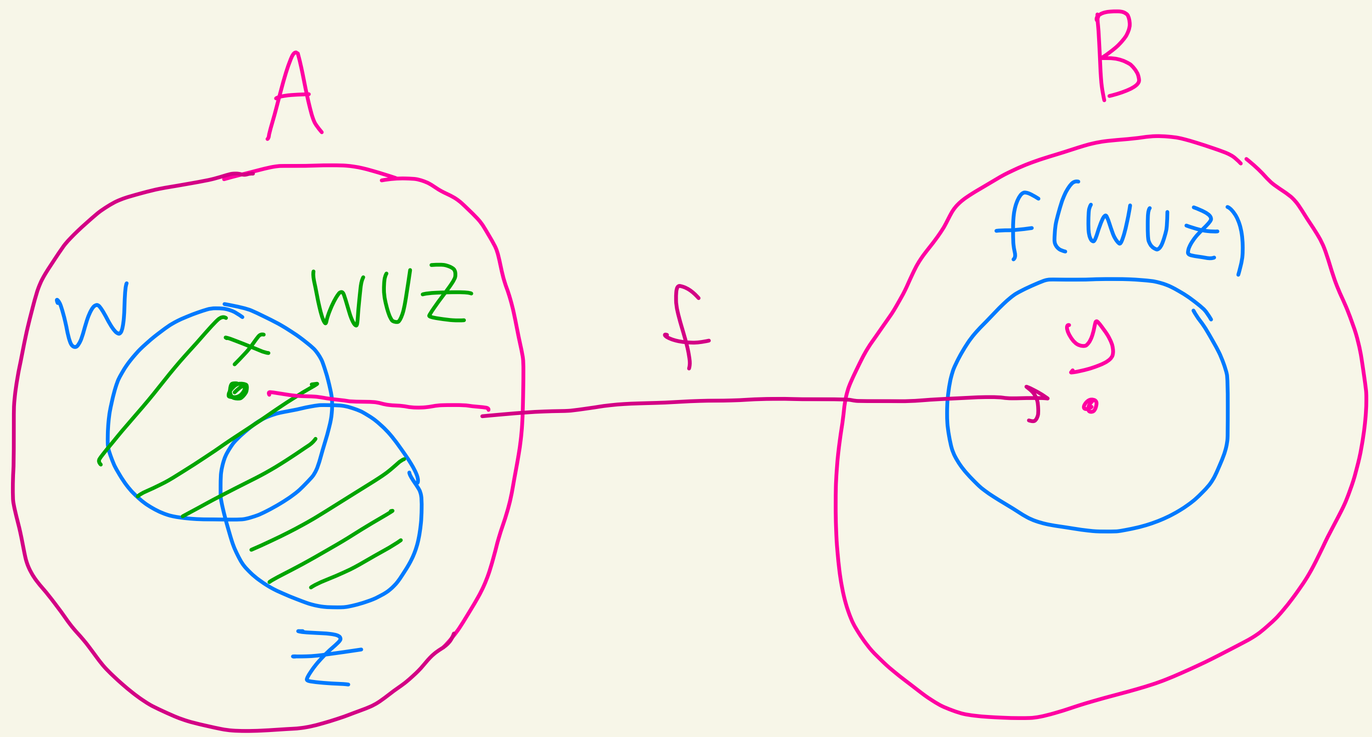
In this example,

$f(W \cap Z) = \{4\} \neq \{4, 5\} = f(W) \cap f(Z)$

① We want to show that

$$f(W \cup Z) = f(W) \cup f(Z)$$

($\subseteq$): Let $y \in f(W \cup Z)$.

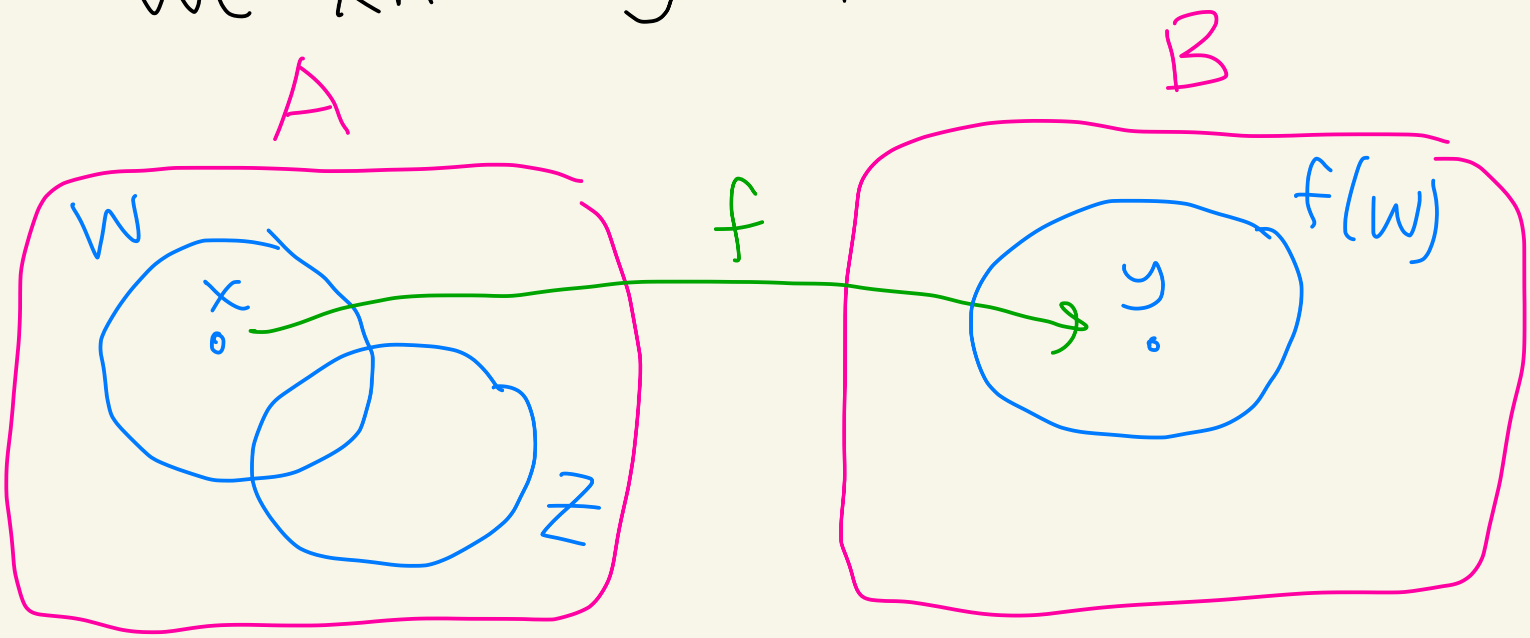


Then there exists $x \in W \cup Z$ where $f(x) = y$.

Since $x \in W \cup Z$ we know $x \in W$ or $x \in Z$.

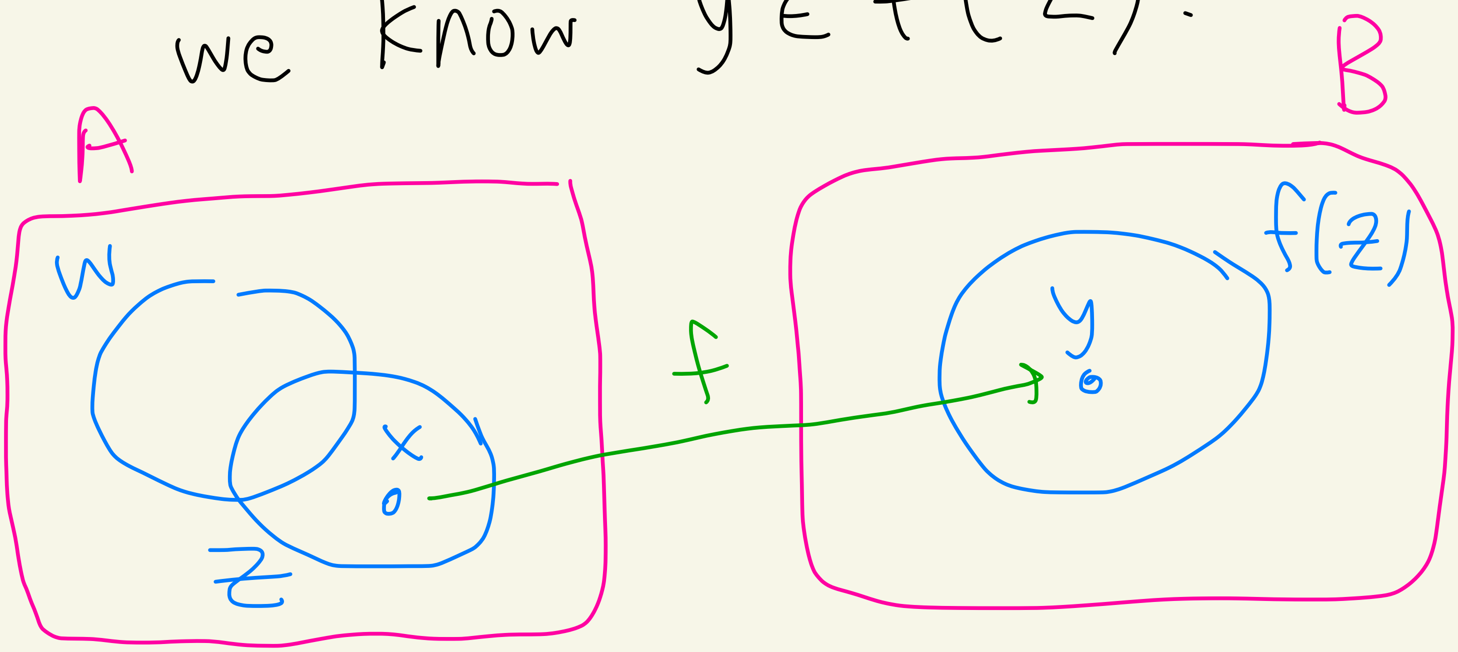
Case 1: Suppose $x \in W$.

Then since $x \in W$ and $f(x) = y$ we know $y \in f(W)$



Case 2: Suppose $x \in Z$.

Then since $x \in Z$ and $f(x) = y$ we know $y \in f(Z)$.



So either $y \in f(W)$ or $y \in f(Z)$ from the two cases above.

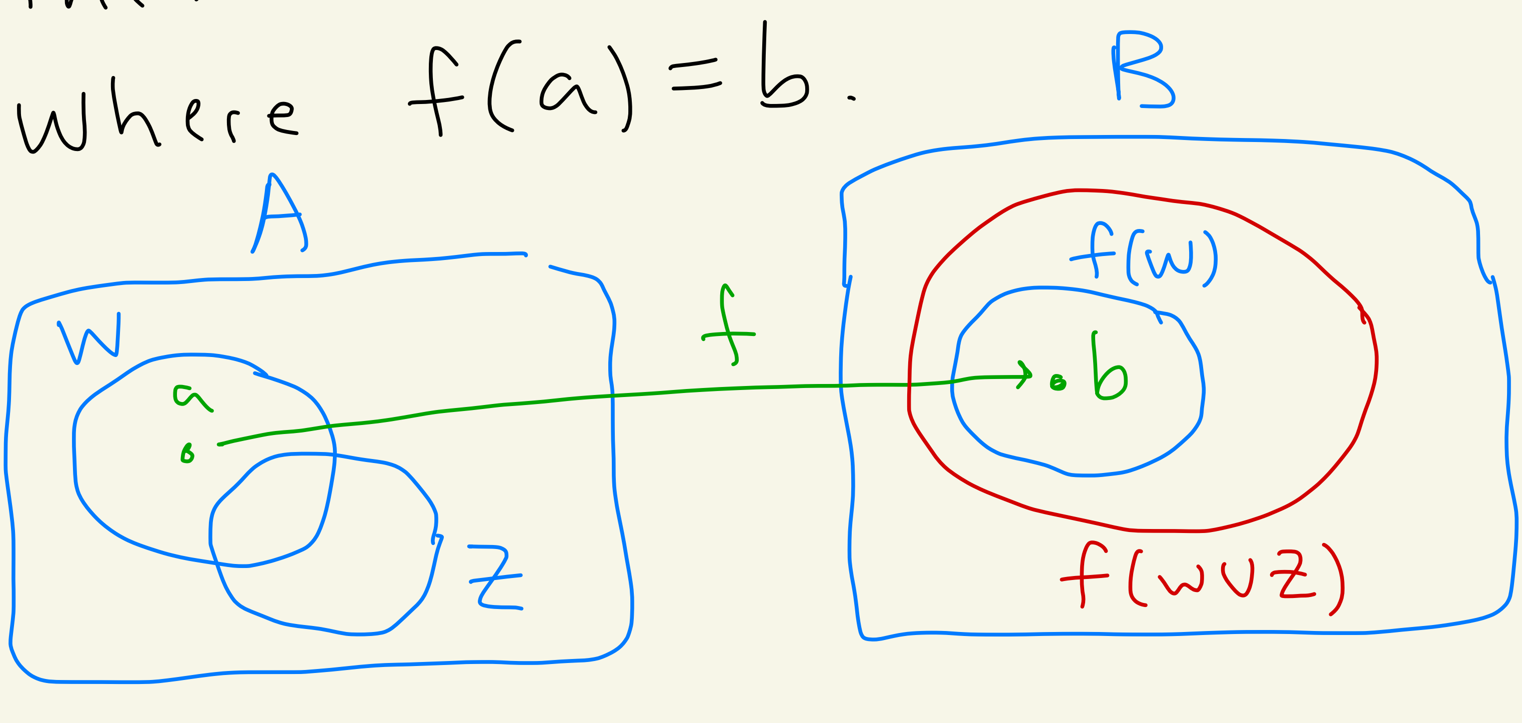
Thus, $y \in f(W) \cup f(Z)$.

($\supseteq$): Let $b \in f(W) \cup f(Z)$.

Then, $b \in f(W)$ or $b \in f(Z)$.

<u>Case 1</u>: Suppose $b \in f(W)$.

Then there exists $a \in W$ where $f(a) = b$.



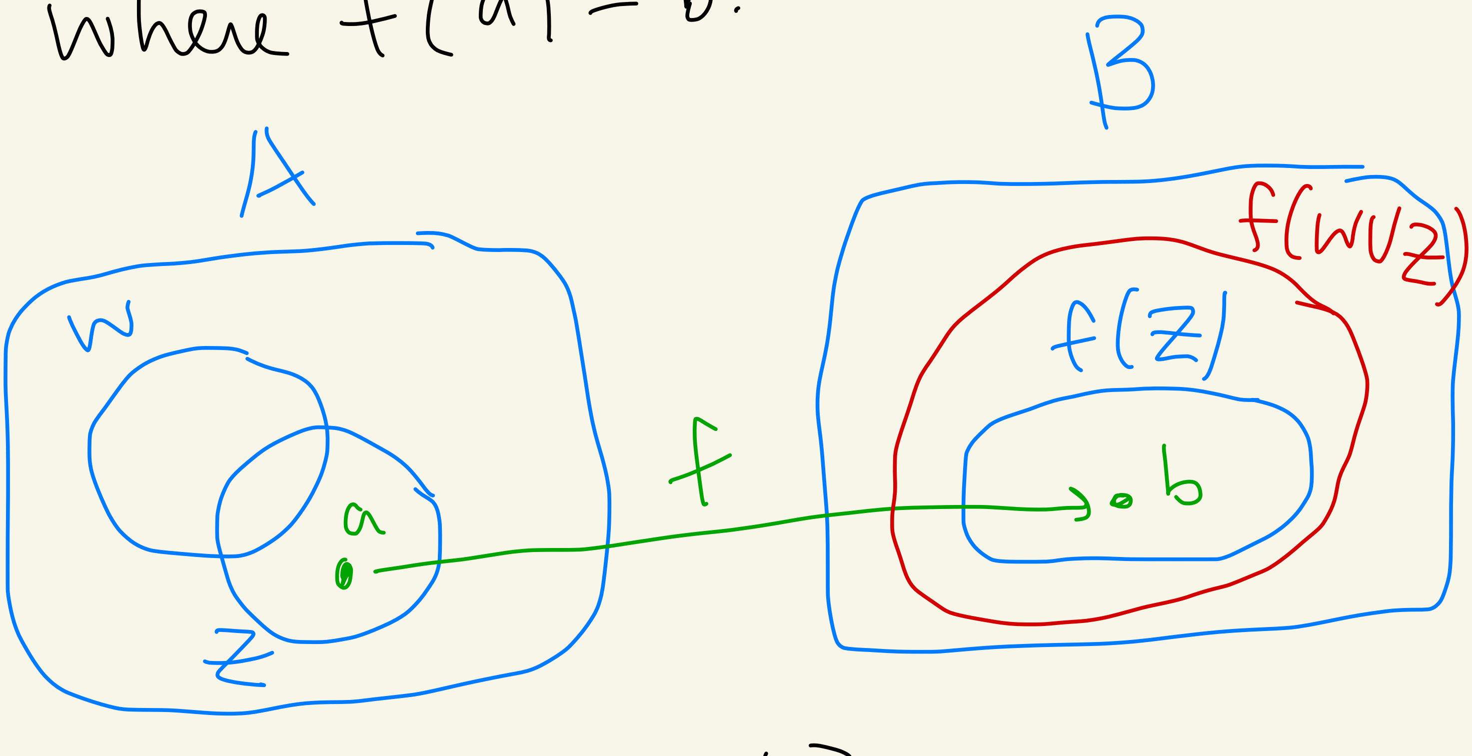
But $a \in W \subseteq W \cup Z$.

So, $a \in W \cup Z$ and $f(a) = b$.

Thus, $b \in f(W \cup Z)$.

<u>Case 2:</u> Suppose $b \in f(Z)$.

Then there exists $a \in Z$ where $f(a) = b$.



But $a \in Z \subseteq W \cup Z$.

So, $a \in W \cup Z$ and $f(a) = b$.

Thus, $b \in f(W \cup Z)$.

Therefore, in either case 1 or case 2 we get $b \in f(W \cup Z)$.

Thus, $f(W) \cup f(Z) \subseteq f(W \cup Z)$.

By, $(\subseteq)$ and $(\supseteq)$ we get

$$f(W \cup Z) = f(W) \cup f(Z).$$

$\blacksquare$