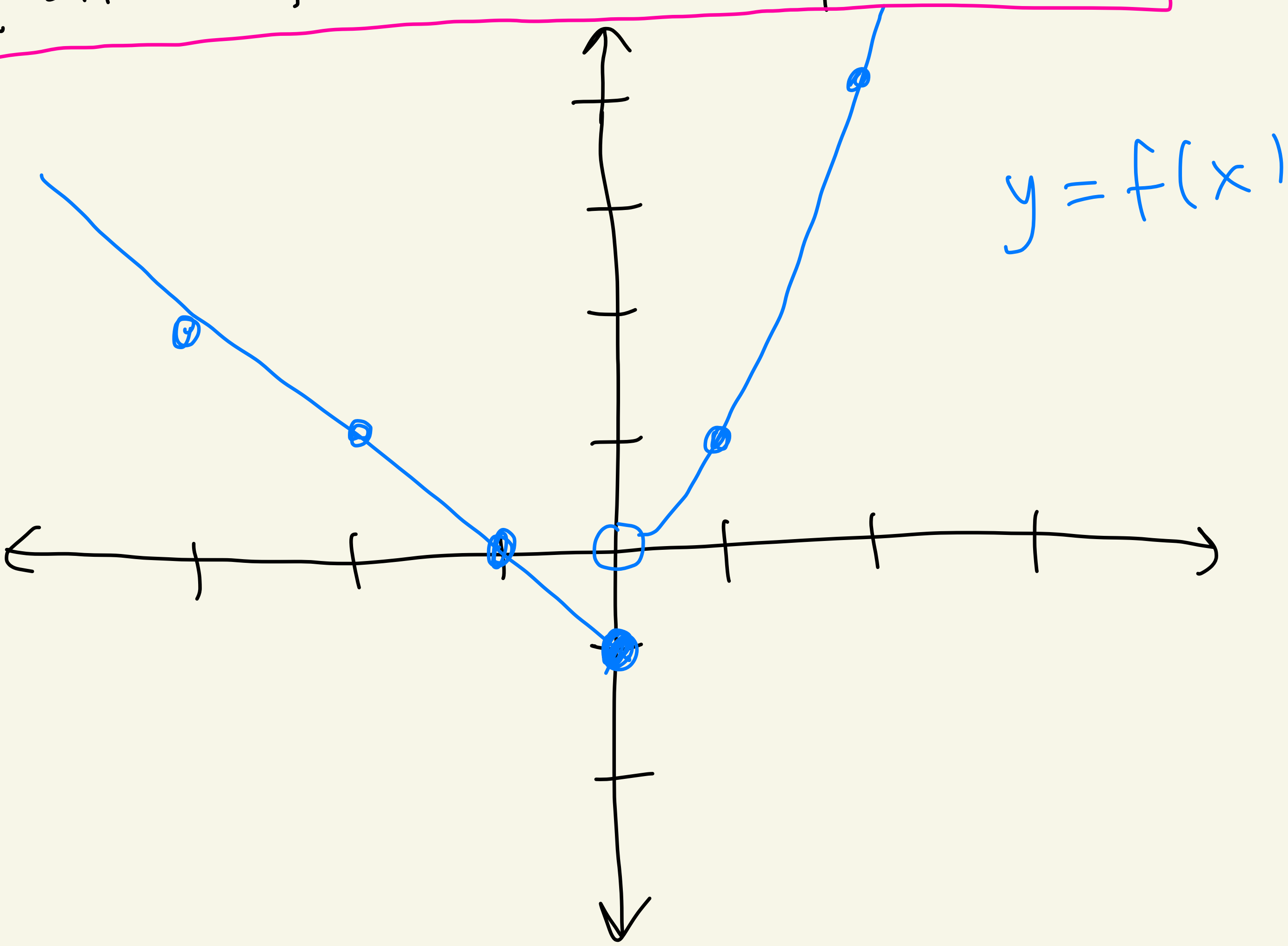
Math 3450

4/23/24

(Like practice test problem)

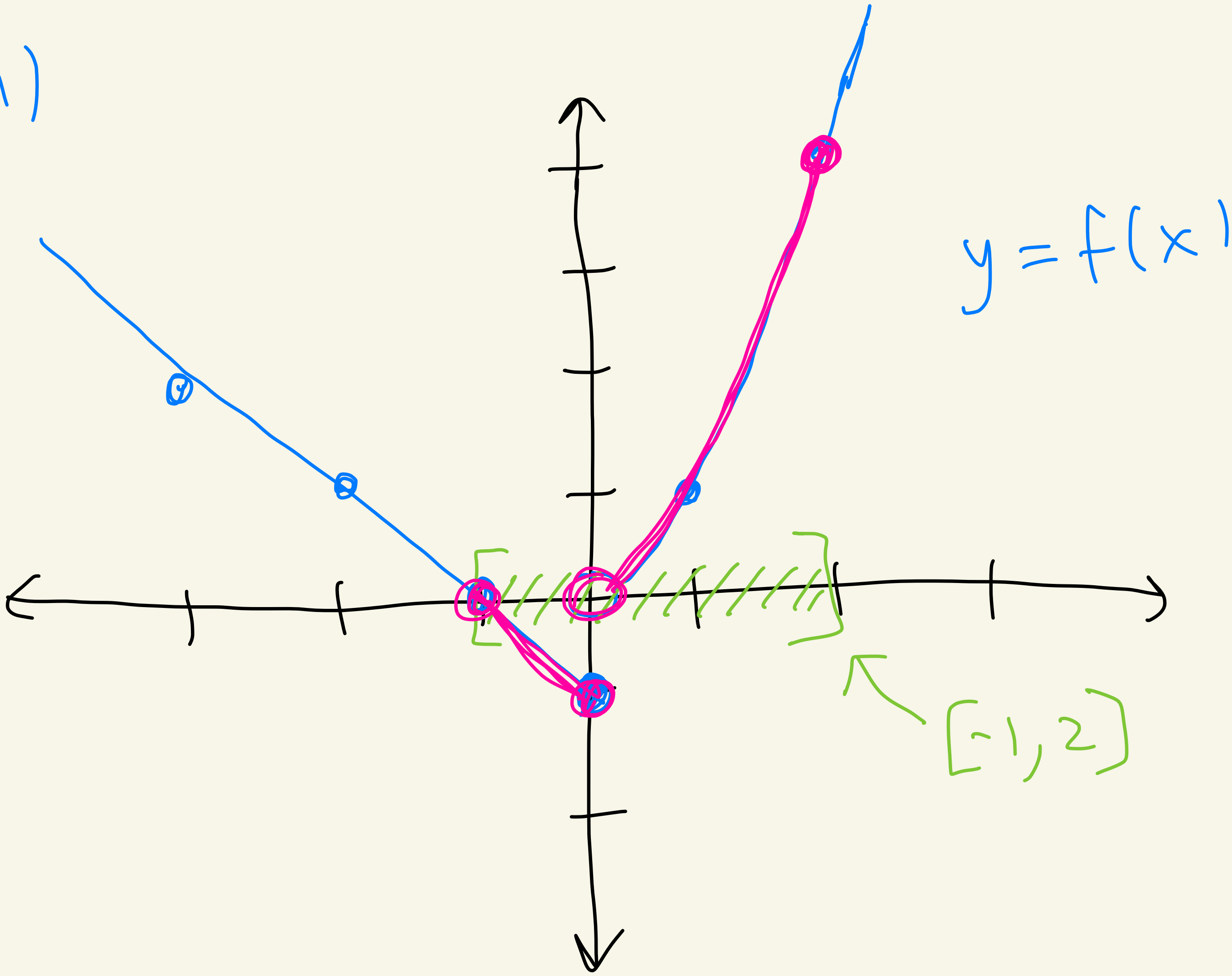


$$f(x) = \begin{cases} -1-x & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$$

Calculate: (a) $f([-1,2])$

(b) $f^{-1}([-3,1])$

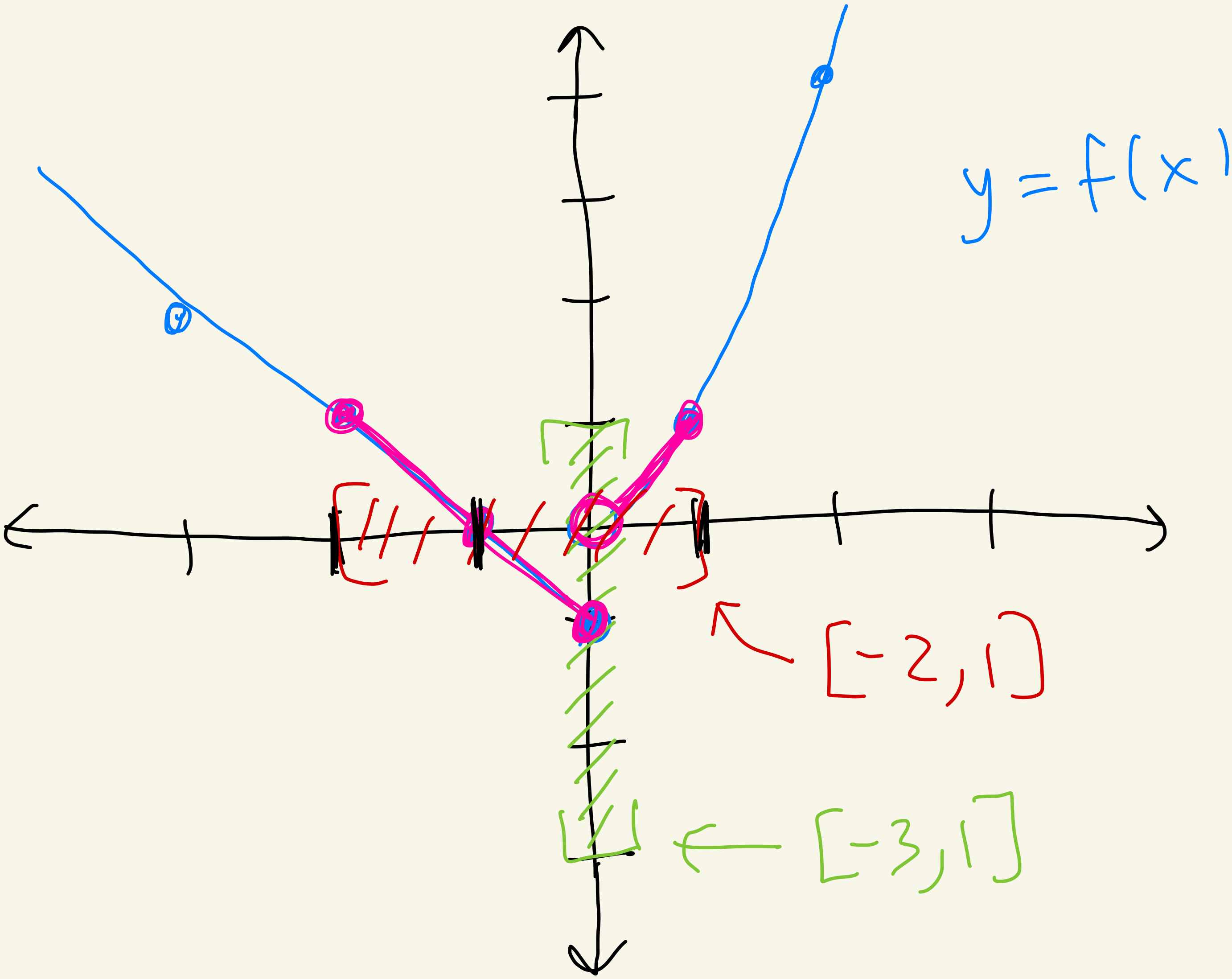
(a)



$$f([-1,2]) = [-1, 4]$$

same

$$[-1, 0] \cup (0, 4]$$



$$f^{-1}([-3,1]) = [-2,1]$$

y values

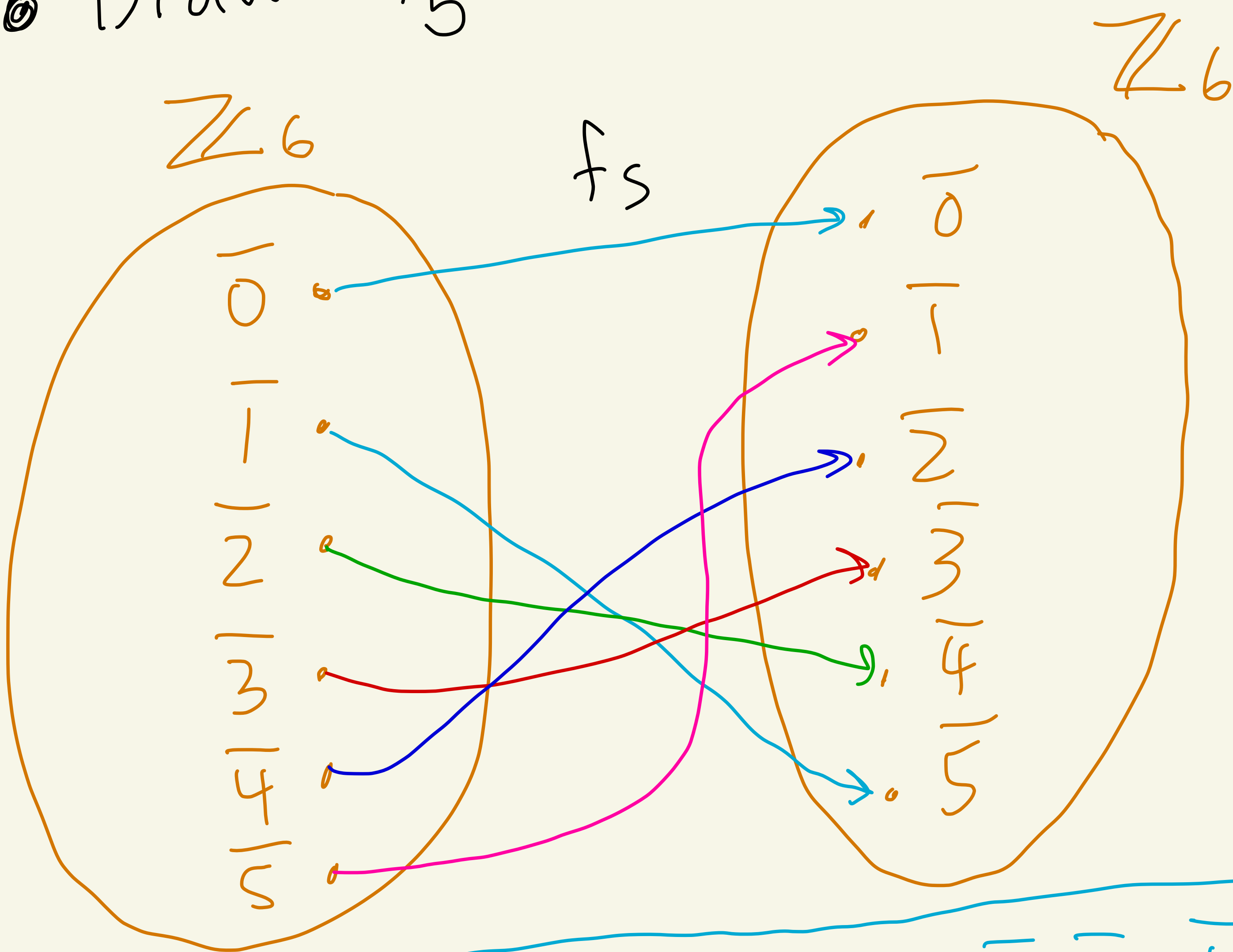
x-values

$\boxed{\text{HW 4 \#4 modified}}$

$a, n \in \mathbb{Z}$, $n \geq 2$, $f_a : \mathbb{Z}_n \to \mathbb{Z}_n$

$f_a(\bar{x}) = \bar{a} \cdot \bar{x}$

- Draw $f_5 : \mathbb{Z}_6 \to \mathbb{Z}_6$



$f_5(\bar{0}) = \bar{5} \cdot \bar{0} = \bar{0}$

$f_5(\bar{1}) = \bar{5} \cdot \bar{1} = \bar{5}$

$f_5(\bar{2}) = \bar{5} \cdot \bar{2} = \overline{10} = \bar{4}$

$\boxed{f_5 \text{ is 1-1 \& onto}}$

Note that $f = f^{-1}$

- **Claim:** Given $f_a : \mathbb{Z}_n \to \mathbb{Z}_n$ where $f_a(\bar{x}) = \bar{a} \cdot \bar{x}$, if there exists $\bar{b} \in \mathbb{Z}_n$ where $\bar{b} \cdot \bar{a} = \bar{1}$, then f_a is a bijection.

Ex:
$\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$
$\bar{a} = \bar{5}$, $\bar{b} = \bar{5}$
$\bar{b} \cdot \bar{a} = \bar{5} \cdot \bar{5} = \overline{25} = \bar{1}$

proof:

(1-1)
Suppose $f_a(\bar{x}_1) = f_a(\bar{x}_2)$ where $\bar{x}_1, \bar{x}_2 \in \mathbb{Z}_n$.

So, $\bar{a} \cdot \bar{x}_1 = \bar{a} \cdot \bar{x}_2$.

Then, $\bar{b} \cdot \bar{a} \cdot \bar{x}_1 = \bar{b} \cdot \bar{a} \cdot \bar{x}_2$

Ex:
$\mathbb{Z}_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$
$\bar{a} = \bar{2}$
$\bar{b} = \bar{3}$
$\bar{3} \cdot \bar{2} = \bar{6} = \bar{1}$

So, $\bar{1}\cdot\bar{x}_1 = \bar{1}\cdot\bar{x}_2$

Thus, $\bar{x}_1 = \bar{x}_2$.

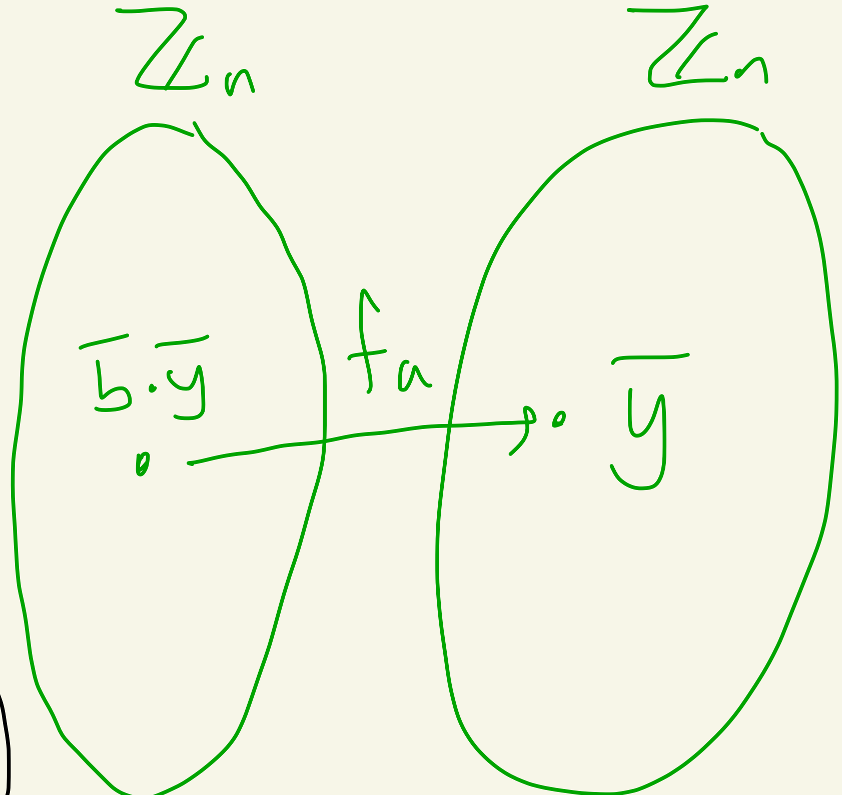
(onto)

Let $\bar{y} \in \mathbb{Z}_n$

Note that
$\bar{b}\cdot\bar{y} \in \mathbb{Z}_n$
and

$$f_a(\bar{b}\cdot\bar{y}) = \bar{a}(\bar{b}\cdot\bar{y})$$
$$= \bar{a}\cdot\bar{b}\cdot\bar{y}$$
$$= \underbrace{\bar{b}\cdot\bar{a}}_{\bar{1}}\cdot\bar{y}$$
$$= \bar{y}.$$

So, f_a is onto.



<u>Scratchwork</u>

Find $\bar{x}$ where $f_a(\bar{x}) = \bar{y}$.

Solve $\bar{a}\cdot\bar{x} = \bar{y}$

$$\underbrace{\bar{b}\cdot\bar{a}}_{\bar{1}}\cdot\bar{x} = \bar{b}\cdot\bar{y}$$

$$\bar{x} = \bar{b}\bar{y}$$

$\boxed{\text{HW 3}}$ 9(e)

$$S = \mathbb{Z} \times (\mathbb{Z} - \{0\})$$
$$= \{(a,b) \mid a, b \in \mathbb{Z},\ b \neq 0\}$$
$$= \{(0,-1), (1,5), (3,-2), \ldots\}$$

Define $(a,b) \sim (c,d)$ iff
$$ad = bc$$

Ex: $(1,1) \sim (2,2) \leftarrow (1)(2) = (1)(2)$

$(2,3) \sim (2,3) \leftarrow (2)(3) = (3)(2)$

$(2,3) \sim (4,6) \leftarrow (2)(6) = (3)(4)$

$$\overline{(2,3)} = \{(2,3), (4,6), (6,9), (8,12), (-2,-3), (-4,-6), \ldots\} = \overline{(4,6)}$$

$= \overline{(6,9)}$

$= \overline{(8,12)}$

$\overline{(1,1)} = \{(1,1), (2,2), (3,3), (-1,-1), \ldots\}$

$= \overline{(2,2)} = \overline{(3,3)}$

(e) Define

$\overline{(a,b)} \oplus \overline{(c,d)} = \overline{(ad+bc, bd)}$

Prove $\oplus$ is well-defined.

$\boxed{\text{Ex: } \overline{(1,1)} \oplus \overline{(2,3)} = \overline{((1)(3)+(1)(2), (1)(3))} = \overline{(5,3)}}$

Proof:

① Let $\overline{(a,b)}, \overline{(c,d)} \in \overbrace{S/\sim}^{\text{set of equiv. classes}}$

Then, $a, b, c, d \in \mathbb{Z}$ and $b \neq 0, d \neq 0$.

Then,

$$\overline{(a,b)} \oplus \overline{(c,d)} = \overline{(ad+bc, bd)} \in S/\sim$$

because $ad+bc \in \mathbb{Z}$, $bd \in \mathbb{Z}$

and $bd \neq 0$ (because $b \neq 0, d \neq 0$).

2 Suppose $\overline{(a,b)} = \overline{(x,y)}$

and $\overline{(c,d)} = \overline{(w,z)}$.

We need to show that

$$\overline{(a,b)} \oplus \overline{(c,d)} = \overline{(ad+bc, bd)}$$

and $\overline{(x,y)} \oplus \overline{(w,z)} = \overline{(xz+yw, yz)}$

are equal.

Since $\overline{(a,b)} = \overline{(x,y)}$ we get $\boxed{ay = bx}$.

Since $\overline{(c,d)} = \overline{(w,z)}$ we get $\boxed{cz = dw}$.

Thus,

$$\begin{aligned}(ad+bc)(yz) &= adyz + bcyz \\ &= (ay)(dz) + (cz)(by) \\ &= (bx)(dz) + (dw)(by) \\ &= (xz)(bd) + (yw)(bd) \\ &= (bd)(xz+yw).\end{aligned}$$

So, $\overline{(ad+bc,\, bd)} = \overline{(xz+yw,\, yz)}$

By ① & ②, $\oplus$ is well-defined.

$\blacksquare$

Practice test #3

$f : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$, $f(m,n) = (m+n, n^3)$

$g : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$, $g(m,n) = (2m+1, n)$

(a)

$g(0,1) = (1,1)$

$(g\circ f)(1,1) = g(f(1,1)) = g(2,1) = (5,1)$

(b)

$$\begin{aligned}(g\circ f)(m,n) &= g(f(m,n)) \\ &= g(m+n, n^3) \\ &= (2(m+n)+1, n^3) \\ &= (2m+2n+1, n^3)\end{aligned}$$

(c) Show that g is 1-1.

Suppose $g(m,n) = g(a,b)$.

Then, $(2m+1, n) = (2a+1, b)$.

Thus, $2m+1 = 2a+1$ and $\boxed{n = b}$.

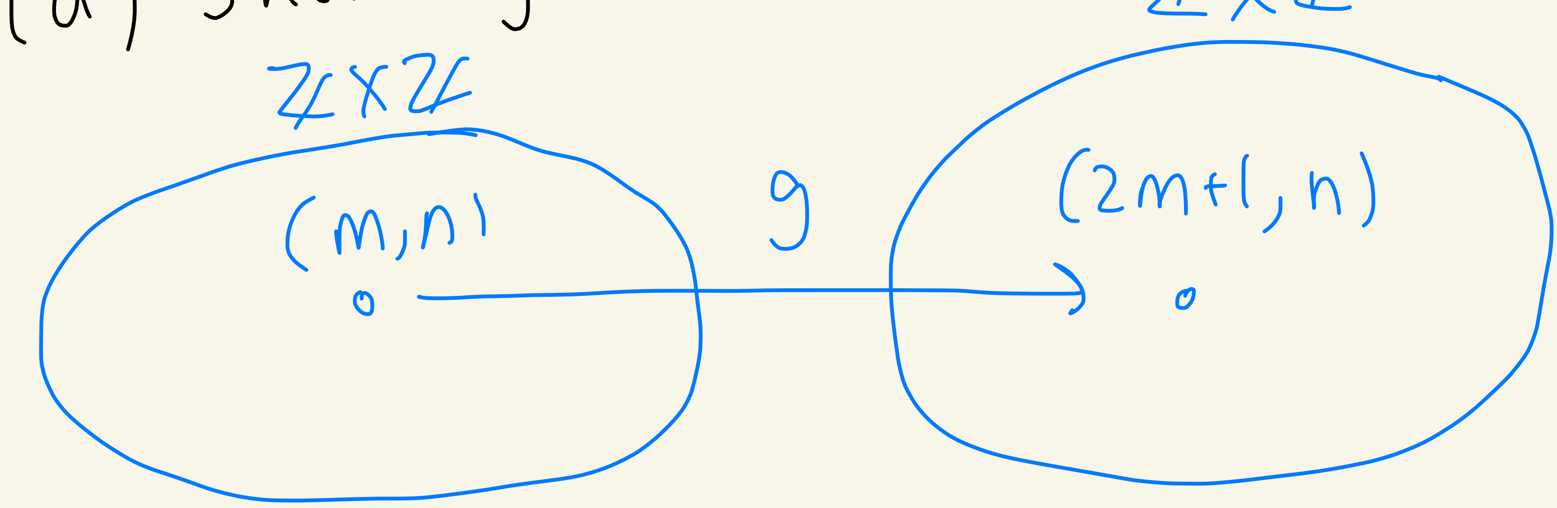
Solving $2m+1 = 2a+1$ gives $2m = 2a$ and then $\boxed{m = a}$.

So, $m = a$ and $n = b$.

Thus, $(m,n) = (a,b)$.

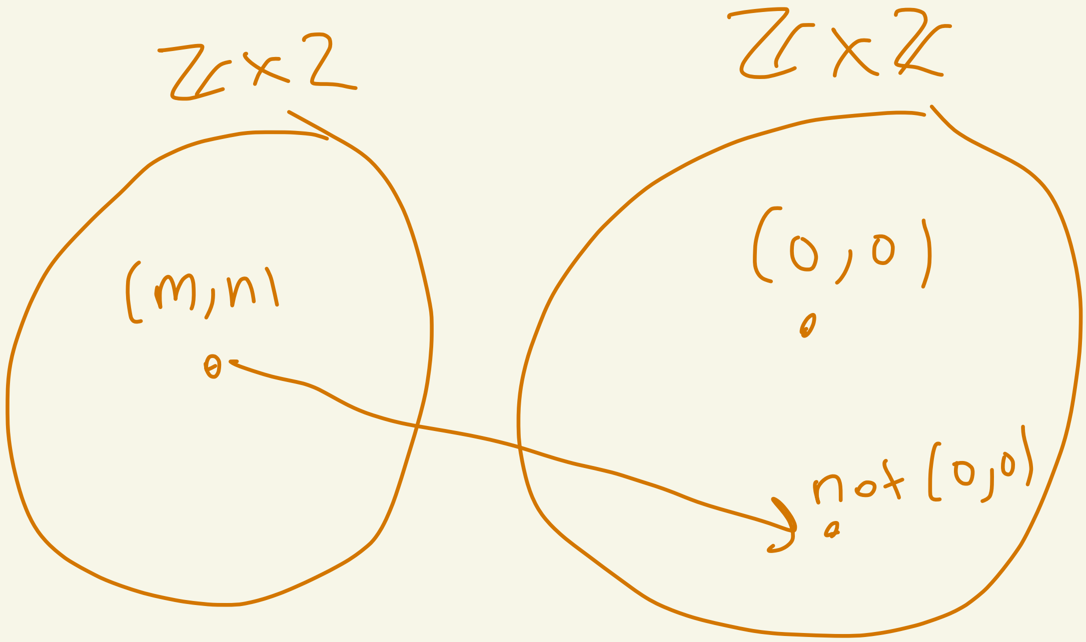
So, g is 1-1.

(d) Show g is not onto.



Let's show $(0,0)$ is not in the range of g.

For $(0,0)$ to be in the range of g you would need



$(m,n) \in \mathbb{Z} \times \mathbb{Z}$ with $g(m,n) = (0,0)$ which is $(2m+1, n) = (0,0)$,

So, you would need $2m+1 = 0$

But that implies $m = \frac{-1}{2}$, which is not an integer.

So, there is no (m,n) with

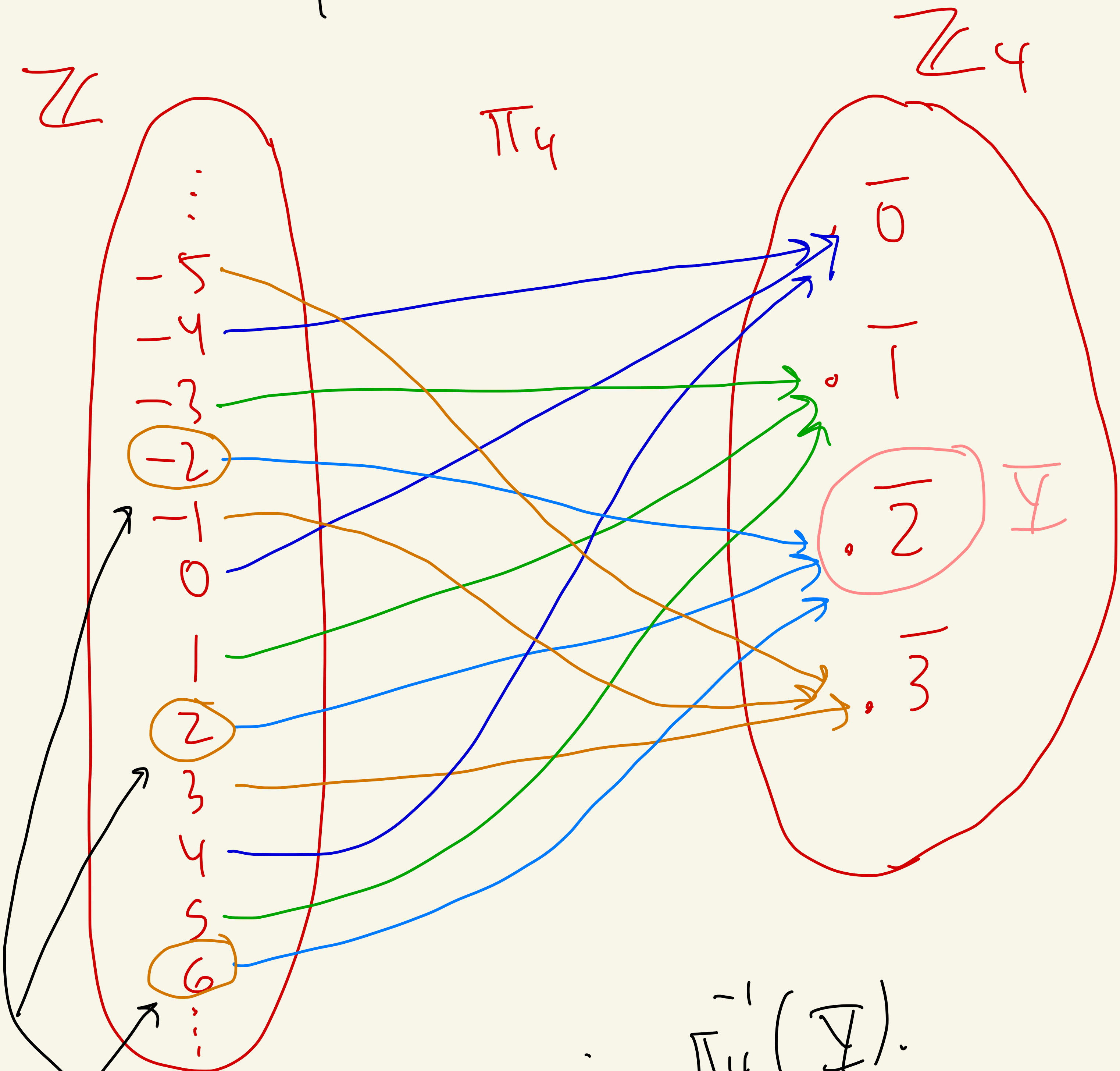
$$g(m,n) = (0,0).$$

Practice Test

4(a) $\pi_4 : \mathbb{Z} \to \mathbb{Z}_4$

$\pi_4(x) = \overline{x}$

Calculate $\pi_4^{-1}(Y)$ where $Y = \{\overline{2}\}$



these are in $\pi_4^{-1}(Y)$.

<u>Prove:</u> $\pi_4^{-1}(Y) = \{4k+2 \mid k \in \mathbb{Z}\}$

> <u>Recall:</u>
> $x \in f^{-1}(W)$ means $f(x) \in W$

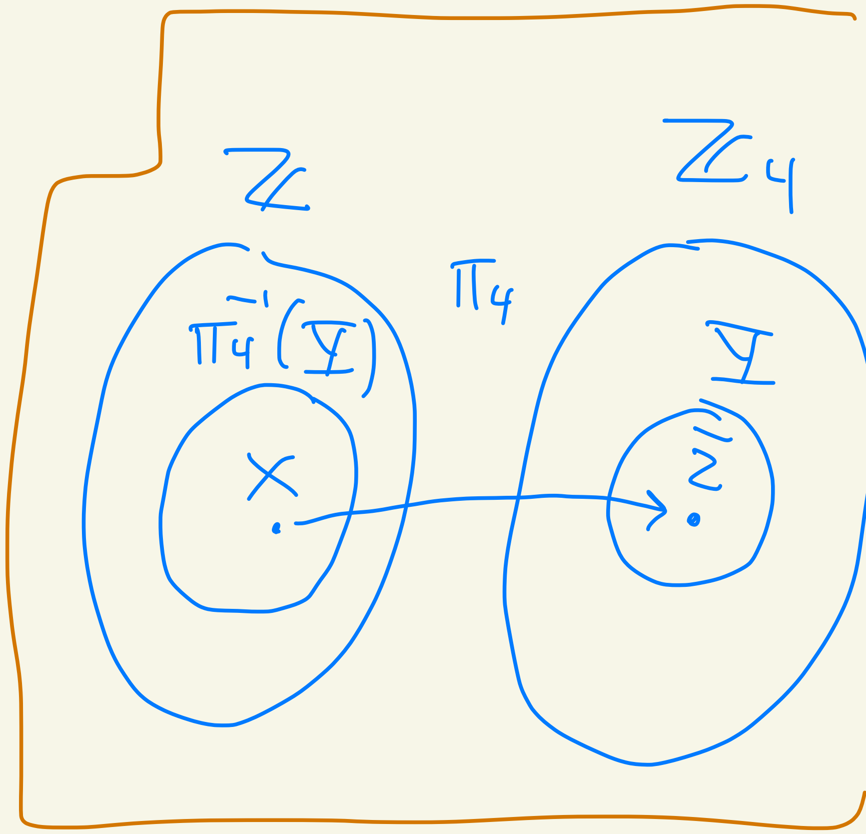
<u>proof:</u>

$(\subseteq)$: Let $x \in \pi_4^{-1}(Y)$

So, $\pi_4(x) \in Y$

So, $\pi_4(x) = \overline{2}$

Then, $\overline{x} = \overline{2}$ in $\mathbb{Z}_4$.



Thus, $x \equiv 2 \pmod 4$

So, $4 \mid (x-2)$.

Then, $x - 2 = 4l$ for some $l \in \mathbb{Z}$.

Thus, $x = 4l + 2$.

So, $x \in \{4k+2 \mid k \in \mathbb{Z}\}$.

(2): Suppose $x \in \{4k+2 \mid k \in \mathbb{Z}\}$

Then, $x = 4t+2$ where $t \in \mathbb{Z}$

Then,

$$\pi_4(x) = \overline{x} = \overline{4t+2}$$

$$= \overline{4}\cdot\overline{t} + \overline{2}$$

$$= \overline{0}\cdot t + \overline{2}$$

$$= \overline{2} \in Y$$

$\overline{0} = \overline{4}$ in $\mathbb{Z}_4$

$Y = \{\overline{2}\}$

Thus, $x \in \pi_4^{-1}(Y)$.

$\blacksquare$

Another way:

$$\pi_4^{-1}(Y) = \{x \in \mathbb{Z} \mid \overbrace{\pi_4(x) = \overline{2}}^{\pi_4(x) \in Y}\}$$

$$= \{x \in \mathbb{Z} \mid \overline{x} = \overline{2}\}$$

$$= \{x \in \mathbb{Z} \mid x \equiv 2 \pmod 4\}$$

$$= \{x \in \mathbb{Z} \mid 4 \mid (x-2)\}$$

$$= \{x \in \mathbb{Z} \mid x - 2 = 4k \text{ for some } k \in \mathbb{Z}\}$$

$$= \{4k+2 \mid k \in \mathbb{Z}\}$$