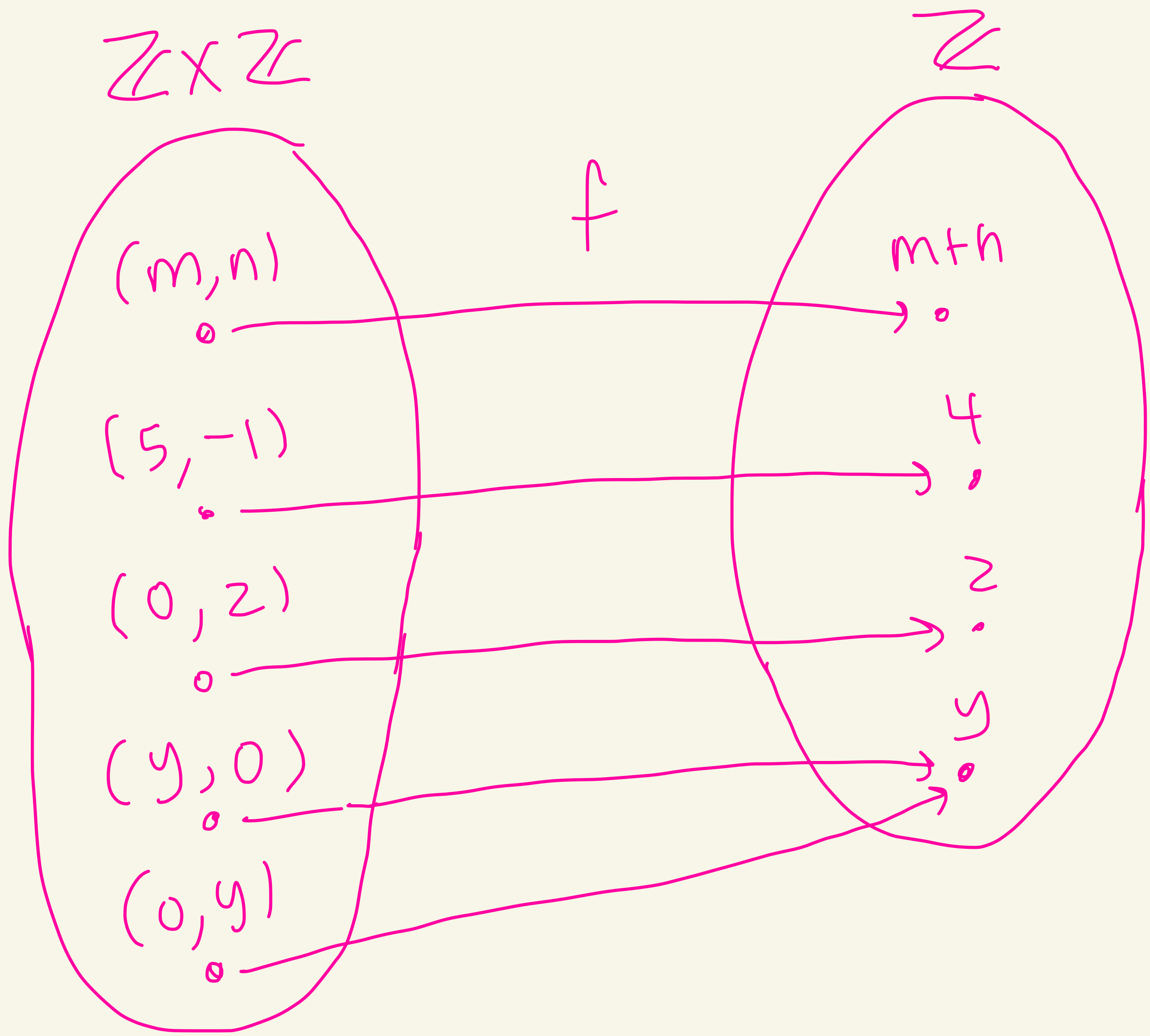
Math 3450
5/9/24

Test 2

3(d) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$$f(m,n) = m+n$$

Show f is onto



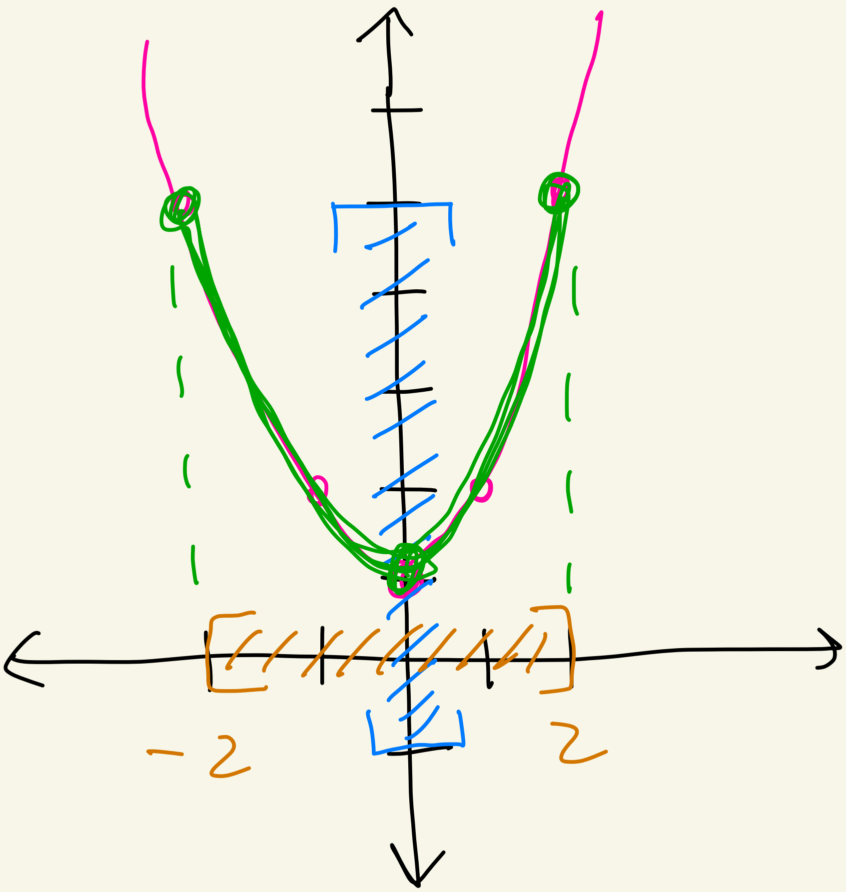
proof: Let $y \in \mathbb{Z}$.
Then, $(y,0) \in \mathbb{Z} \times \mathbb{Z}$ and
$$f(y,0) = y + 0 = y.$$
So, f is onto. $\blacksquare$

$\boxed{\text{Test 2}}$

② $f(x) = x^2 + 1$

$f^{-1}([-1,5])$

$= [-2,2]$

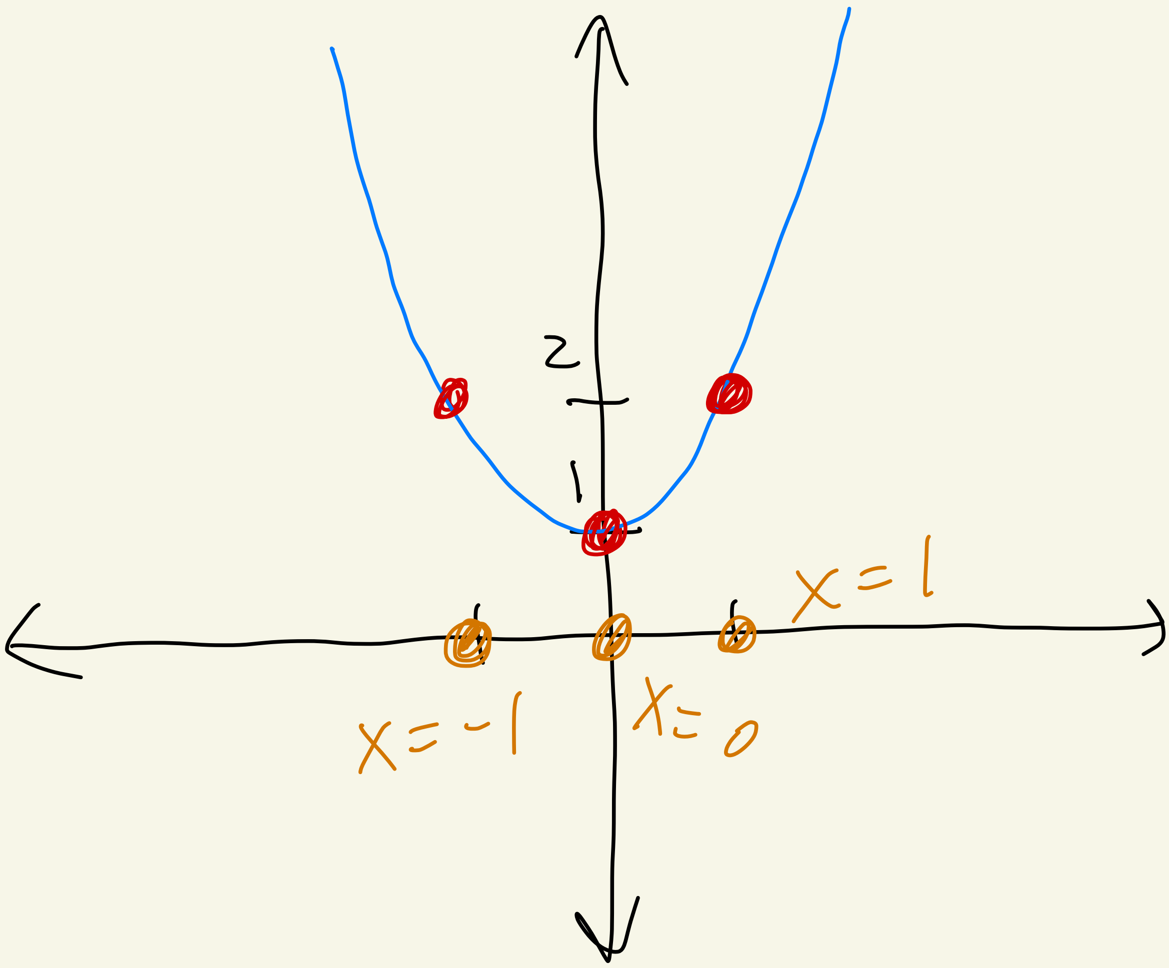


$f^{-1}(\{0,1,2\}) = \{0,1,-1\}$

There are no $x \in \mathbb{R}$ with $x^2+1=0$.

The $x \in \mathbb{R}$ with $x^2+1=1$ is $x=0$

The $x \in \mathbb{R}$ with $x^2+1=2$ are $x=\pm 1$



Ex: Define $\sim$ on $\mathbb{Z}$ where $x \sim y$ means $\underbrace{x+y \text{ is even.}}_{\text{same as: } 2 \mid (x+y)}$

Prove $\sim$ is an equivalence relation.

proof:

(reflexive)

Let $x \in \mathbb{Z}$.

Need to show that $x \sim x$.

We have $x + x = 2x$.

So, $2 \mid (x+x)$.

Thus, $x \sim x$.

(symmetric)

Let $x, y \in \mathbb{Z}$ and assume $x \sim y$.

Since $x \sim y$ we know $2 \mid (x+y)$.

Then, $x+y=2k$ where $k \in \mathbb{Z}$.
So, $y+x=2k$.

Thus, $2 \mid (y+x)$.

So, $y \sim x$.

(<u>transitive</u>)

Let $x, y, z \in \mathbb{Z}$ and assume that $x \sim y$ and $y \sim z$.

We need to show that $x \sim z$.

Since $x \sim y$ and $y \sim z$ we know $2 \mid (x+y)$ and $2 \mid (y+z)$.

So, $x+y=2k$ and $y+z=2\ell$

where $k, l \in \mathbb{Z}$.

Then, by adding we get

$$x + 2y + z = 2k + 2l.$$

So,

$$x + z = 2k + 2l - 2y$$

Thus,

$$x + z = 2(\underbrace{k + l - y}_{\text{this is an integer}}).$$

So,

$$2 \mid (x + z)$$

Thus, $x \sim z$. $\blacksquare$

Another way for transitivity:

$$x + z = (2k - y) + (2\ell - y)$$

$$= 2(k + \ell - y)$$

sub: $x = 2k - y$

$z = 2\ell - y$

Test 1

6C $S = \mathbb{Z} \times (\mathbb{Z} - \{0\})$

$= \{(a,b) \mid a,b \in \mathbb{Z}, b \neq 0\}$

$= \{(1,2), (-1,-3), (1,-5),$
$(0,2), \ldots\}$

Define $(a,b) \sim (c,d)$

means $ad = bc$.

Prove $\sim$ is an equivalence relation.

proof:

(reflexive)

Let $(a,b) \in S$.

Then, $(a,b) \sim (a,b)$ because $ab = ba$.

(symmetric)

Let $(a,b),(c,d) \in S$.

Assume $(a,b) \sim (c,d)$.

Then, $ad = bc$.

So, $cb = da$.

Thus, $(c,d) \sim (a,b)$.

(transitive)

Let $(a,b),(c,d),(e,f) \in S$.

Assume $(a,b) \sim (c,d)$

and $(c,d) \sim (e,f)$.

Since $(a,b) \sim (c,d)$ we know $ad = bc$.

Since $(c,d) \sim (e,f)$ we know $cf = de$.

Since $(c,d),(e,f) \in S$ we know $d \neq 0$, $f \neq 0$.

Then,

$$af = \underbrace{\left(\frac{bc}{d}\right)}_{\substack{a = \frac{bc}{d} \\ \text{ok, since} \\ d \neq 0}} f = \left(\frac{bf}{d}\right) c = \left(\frac{bf}{d}\right) \underbrace{\left(\frac{de}{f}\right)}_{\substack{c = \frac{de}{f} \\ \text{ok, since} \\ f \neq 0}}$$

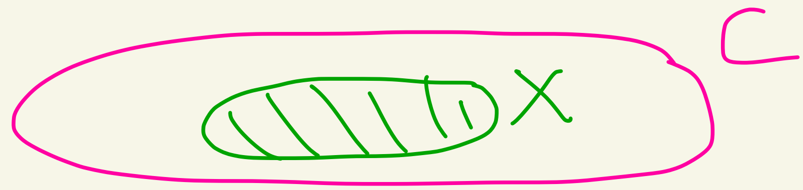
$$= be$$

So, $(a,b) \sim (e,f)$.

$\blacksquare$

HW 2
14(a)

$X \in \mathcal{P}(C)$
means: $X \subseteq C$

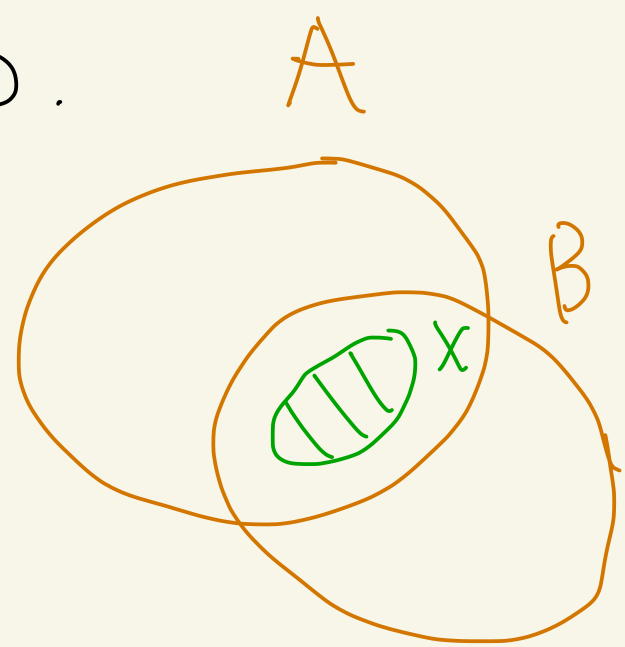


Show $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

<u>proof:</u>

$(\subseteq)$: Let $X \in \mathcal{P}(A \cap B)$

Then, $X \subseteq A \cap B$.



So, $X \subseteq A$
and $X \subseteq B$.

Then, $X \in \mathcal{P}(A)$
and $X \in \mathcal{P}(B)$

So, $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

Thus, $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$

$(\supseteq)$: Let $Y \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

Then, $Y \in \mathcal{P}(A)$ and $Y \in \mathcal{P}(B)$.

So, $Y \subseteq A$ and $Y \subseteq B$.

Thus, $Y \subseteq A \cap B$.

So, $Y \in \mathcal{P}(A \cap B)$.

Thus, $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

By $(\subseteq)$ and $(\supseteq)$ we have

$$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B). \quad \blacksquare$$

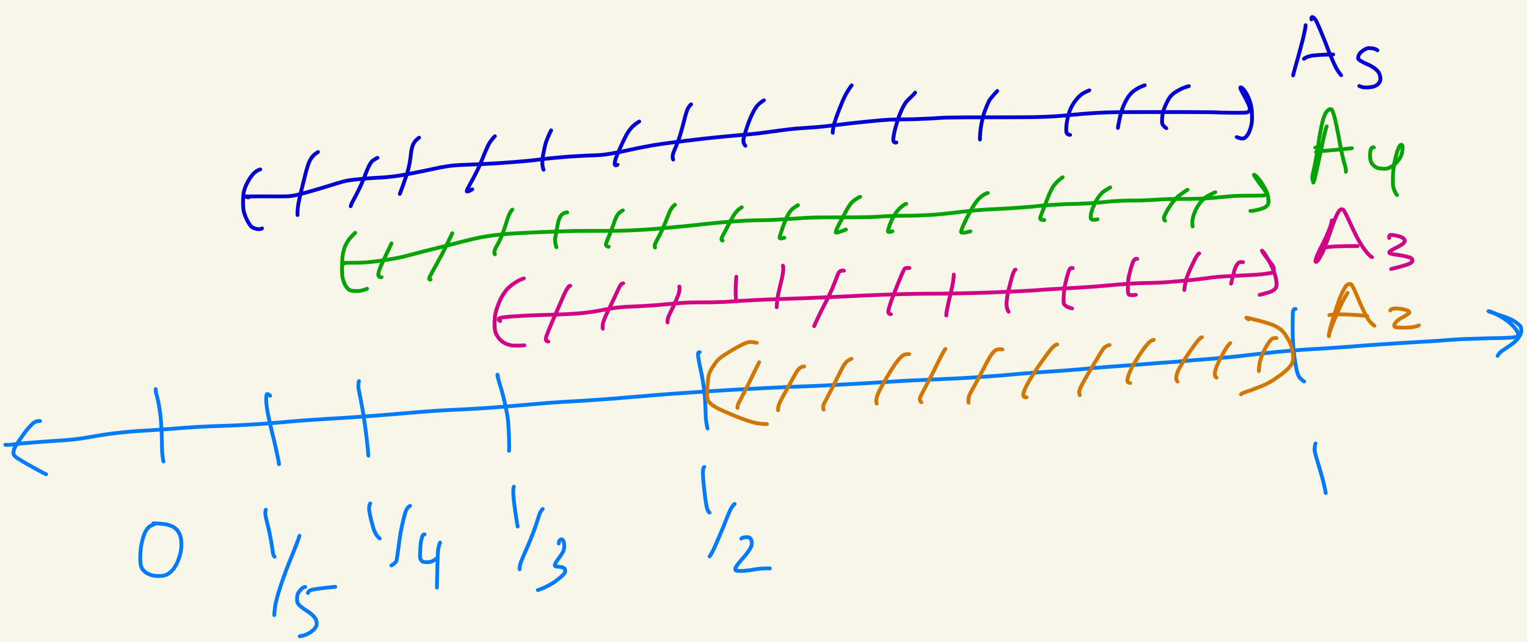
$\boxed{\text{Hw 2}}$

$\boxed{9(b)}$ $A_n = \left(\frac{1}{n}, 1\right)$

Calculate $\bigcup_{n=2}^{\infty} A_n$ and $\bigcap_{n=2}^{\infty} A_n$

$A_2 = \left(\frac{1}{2}, 1\right)$ $A_4 = \left(\frac{1}{4}, 1\right)$

$A_3 = \left(\frac{1}{3}, 1\right)$ $A_5 = \left(\frac{1}{5}, 1\right)$



$$\bigcup_{n=2}^{\infty} A_n = (0, 1) \qquad \bigcap_{n=2}^{\infty} A_n = \left(\frac{1}{2}, 1\right)$$