Math 4300

10/16/23

# Topic 7 – The plane separation axiom

<u>Def:</u> Let $(\mathcal{P}, \mathcal{L}, d)$ be a metric geometry. Let $S \subseteq \mathcal{P}$.
We say that $S$ is <u>convex</u> if given any two points $P, Q \in S$, then $\overline{PQ} \subseteq S$.

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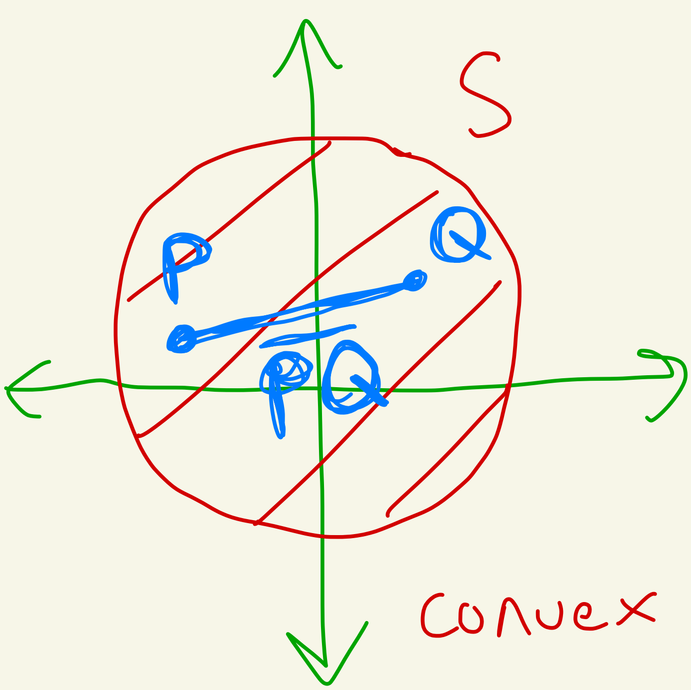
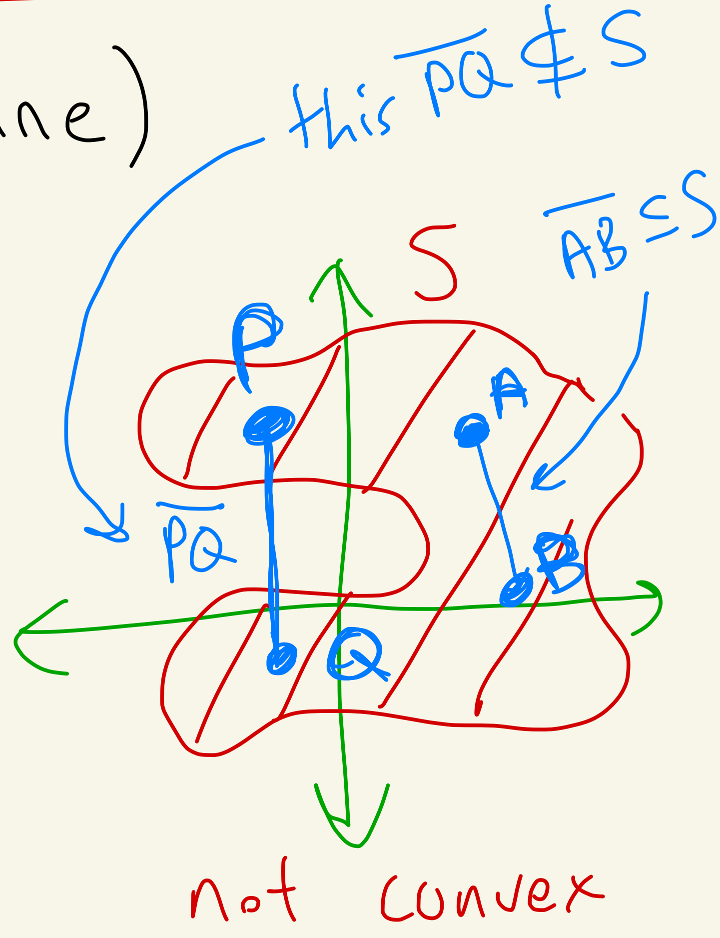
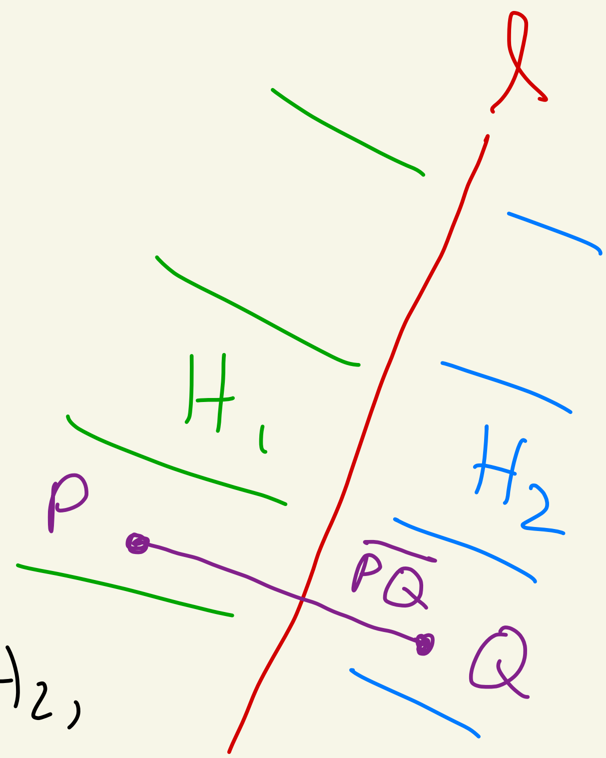
<u>Ex:</u> (Euclidean plane)



**Def:** We say that a metric geometry $(\mathscr{P}, \mathscr{L}, d)$ satisfies the <u>plane separation axiom</u> (PSA) if for every line $\ell \in \mathscr{L}$ there exist two non-empty subsets $H_1 \subseteq \mathscr{P}$ and $H_2 \subseteq \mathscr{P}$ such that




(i) $\mathscr{P} - \ell = H_1 \cup H_2$

(ii) $H_1 \cap H_2 = \emptyset$

(iii) $H_1$ is convex and $H_2$ is convex

(iv) If $P \in H_1$ and $Q \in H_2$, then $\overline{PQ} \cap \ell \neq \emptyset$

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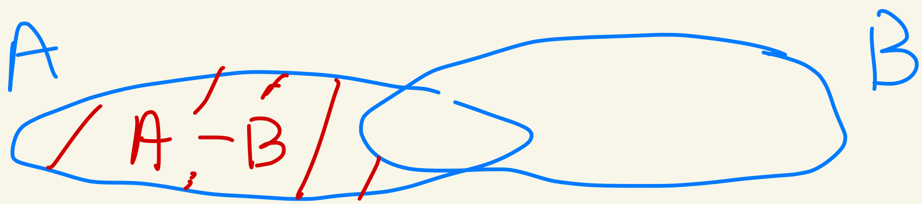
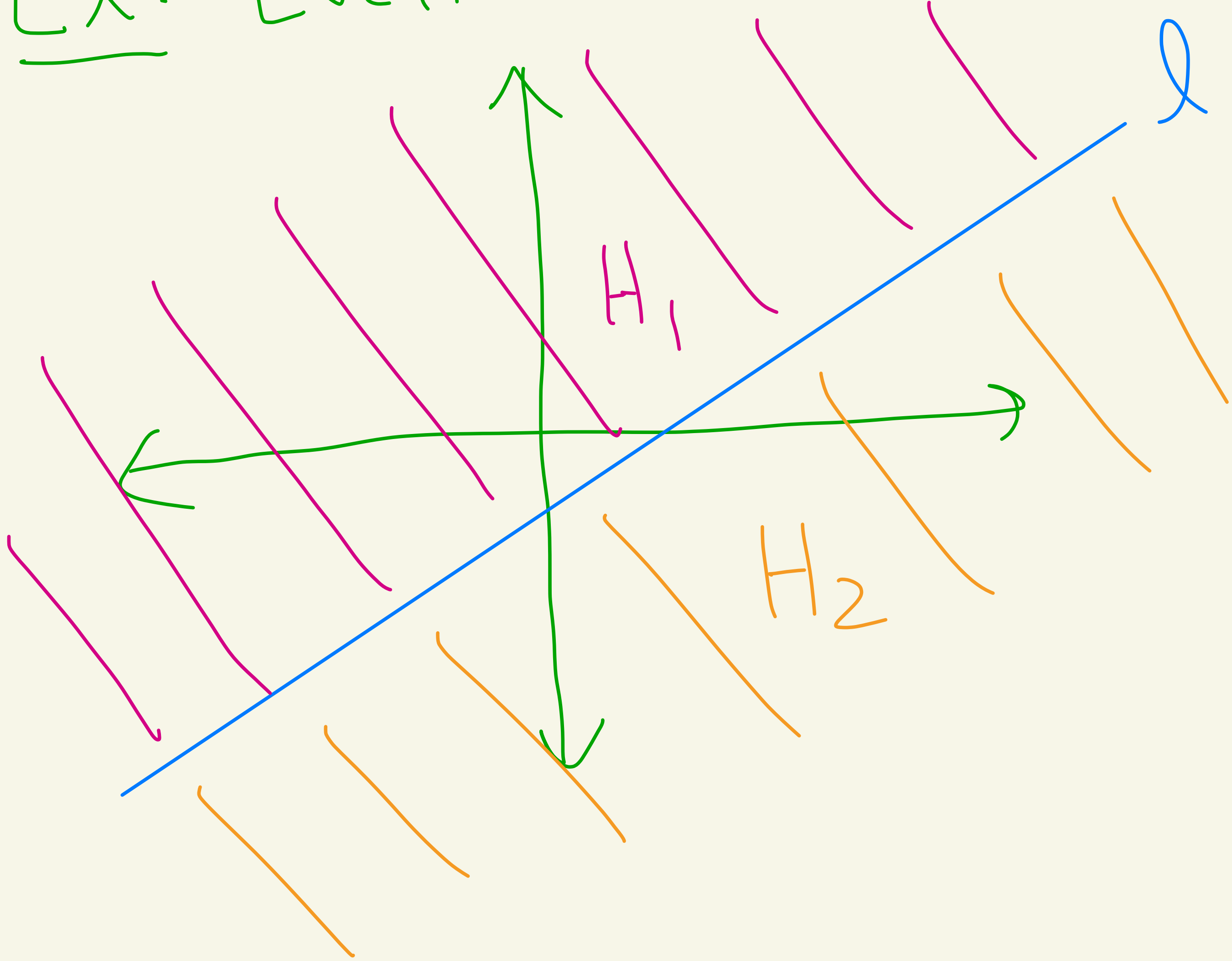
Recall $A - B = \{ x \mid x \in A \text{ and } x \notin B \}$

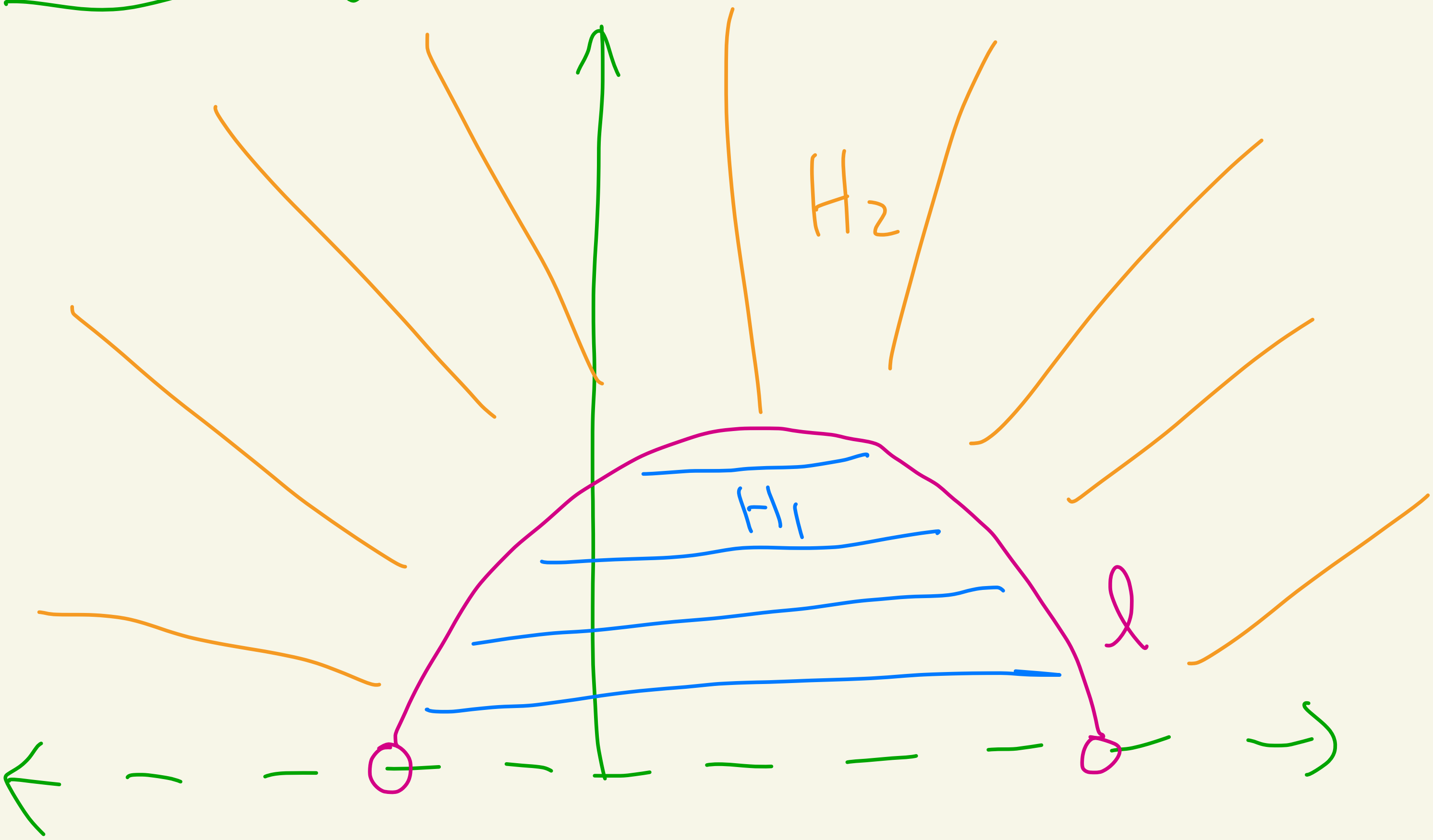


The subsets $H_1$ and $H_2$ are called <u>half-planes determined by $\ell$</u>.

<u>Ex:</u> Euclidean plane



Ex: Hyperbolic plane




We will see later that both the Euclidean plane and the Hyperbolic plane satisfy the plane separation axiom (PSA). For now we prove some general results.

**Theorem** (The half-planes determined by $\ell$ are unique)

Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry that satisfies the PSA. Let $\ell \in \mathscr{L}$.

If $H_1, H_2$ satisfy (i)–(iv) of the PSA for $\ell$ and $H_1', H_2'$ satisfy (i)–(iv) of the PSA for $\ell$,

then either $H_1 = H_1'$ and $H_2 = H_2'$

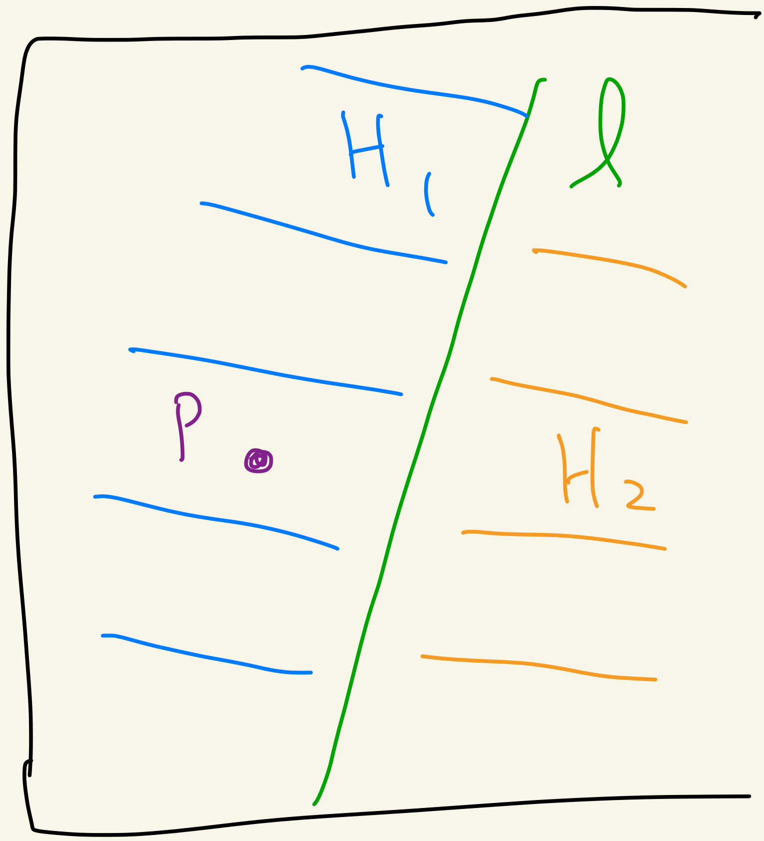
or $H_1 = H_2'$ and $H_2 = H_1'$.

Proof:

Pick some point $P \in H_1$.

By (i) of PSA we know $P \notin \ell$.

Since $H_1'$ and $H_2'$ also satisfy the PSA axioms we have either $P \in H_1'$ or $P \in H_2'$.




Let's assume $P \in H_1'$.

[The case $P \in H_2'$ is similar.] ← Would give $H_1 = H_2'$, $H_2 = H_1'$

Now we will show that

$$H_1 = H_1' \text{ and } H_2 = H_2'.$$

We will first show $H_1 = H_1'$ by showing $H_1 \subseteq H_1'$ and $H_1' \subseteq H_1$.

$\boxed{H_1 \subseteq H_1'}$: Let $Q \in H_1$.

If $Q = P$, then $Q \in H_1'$.

So, suppose $Q \neq P$.

We need to show that $Q \in H_1'$.

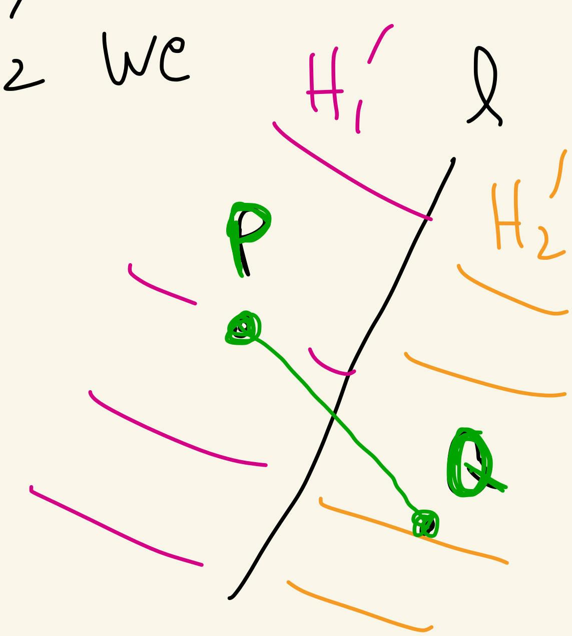
Suppose instead that $Q \notin H_1'$.

Since $\boxed{Q \in H_1}$ we know $Q \notin \ell$.

So since $Q \notin H_1'$ we must have $Q \in H_2'$.

Since $P \in H_1'$ and $Q \in H_2'$ we know $\boxed{\overline{PQ} \cap \ell \neq \emptyset}$ (from PSA (iv)).

But also $P, Q \in H_1$



and $H_1$ is convex,

thus $\overline{PQ} \subseteq H_1$,

which means $\boxed{\overline{PQ} \cap \ell = \phi}$.

Contradiction.

Thus, $Q \in H_1'$.

So, $H_1 \subseteq H_1'$.

A similar argument shows $H_1' \subseteq H_1$. (Try it.)

So, $H_1 = H_1'$.

Then,

$$H_2 = (\mathscr{P} - \ell) - H_1$$

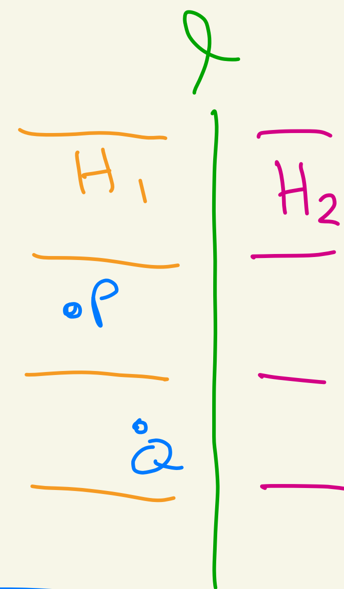
$$= (\mathscr{P} - \ell) - H_1' = H_2'$$

$H_1 = H_1'$

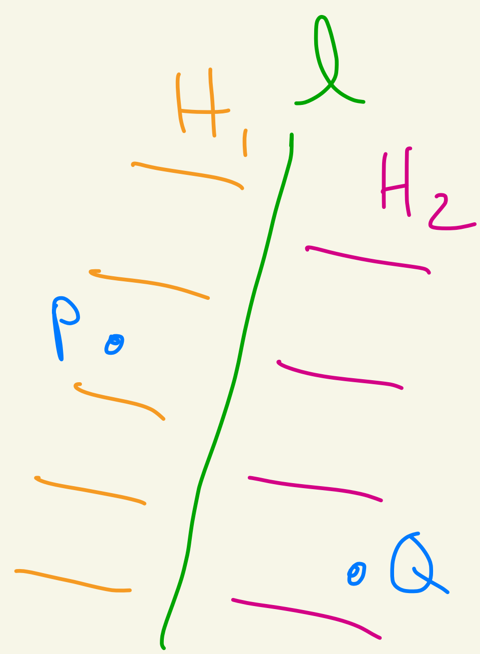
Thus, $H_1 = H_1'$ and $H_2 = H_2'$.

$\blacksquare$

**Def:** Let $(\mathcal{P}, \mathcal{L}, d)$ be a metric geometry that satisfies the PSA. Let $\ell \in \mathcal{L}$ and $H_1, H_2$ be the half-planes determined by $\ell$. Let $P, Q \in \mathcal{P}$.




(i) We say that $P$ and $Q$ lie on the <u>same side of $\ell$</u> if either $P, Q \in H_1$ or $P, Q \in H_2$.




(ii) we say that $P$ and $Q$ lie on <u>opposite sides of $\ell$</u> if either

$P \in H_1$ and $Q \in H_2$

or $P \in H_2$ and $Q \in H_1$

(iii) If $P \in H_1$ then we say that $H_1$ is <u>the side of $\ell$ that contains $P$</u>.