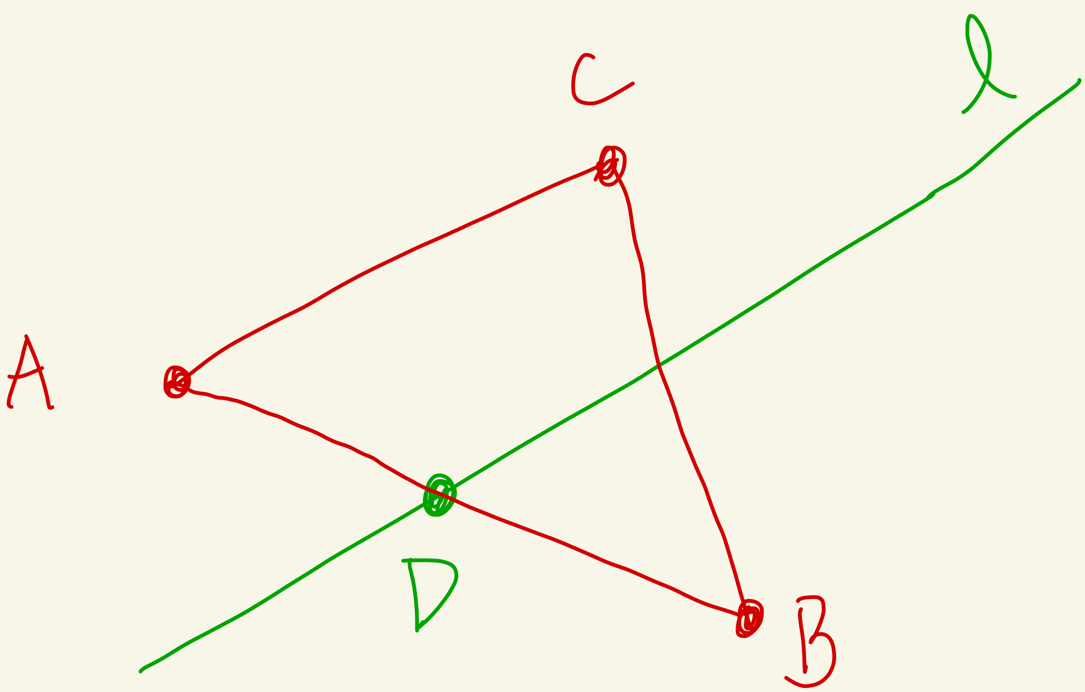
Math 4300
10/25/23

Topic 8 - Pasch Geometries

<u>Def:</u> We say that a metric geometry $(\mathscr{P}, \mathscr{L}, d)$ satisfies the <u>Pasch Postulate</u> (PP) if given a line ℓ, a triangle $\triangle ABC$, and a point D on ℓ with $A-D-B$, then either $\ell \cap \overline{AC} \neq \phi$ or $\ell \cap \overline{BC} \neq \phi$



(picture has $\ell \cap \overline{BC} \neq \phi$)

It turns out that in metric geometries the Pasch Postulate is equivalent to the Plane Seperation axiom.

Millman/Parker (pgs 79-80) they give an example of a metric geometry called the <u>Missing strip plane</u> that doesn't satisfy the PP or PSA.

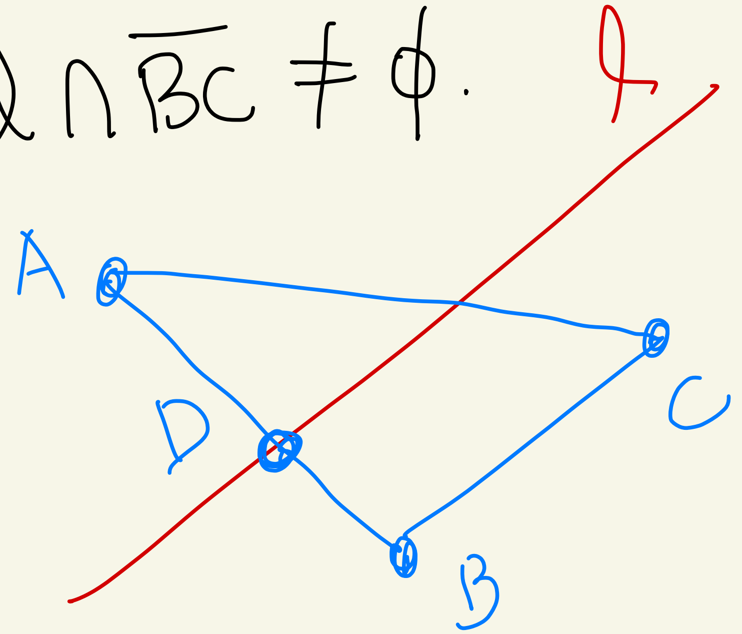
__Theorem (Pasch's Theorem)__

If a metric geometry $(\mathscr{P}, \mathscr{L}, d)$ satisfies PSA, then it satisfies PP.

__Proof:__ Suppose we have a metric geometry $(\mathscr{P}, \mathscr{L}, d)$ that satisfies PSA.

Let $\triangle ABC$ be a triangle, $\ell \in \mathscr{L}$ with $D \in \ell$ and $A-D-B$.

We will show that either $\ell \cap \overline{AC} \neq \phi$ or $\ell \cap \overline{BC} \neq \phi$.



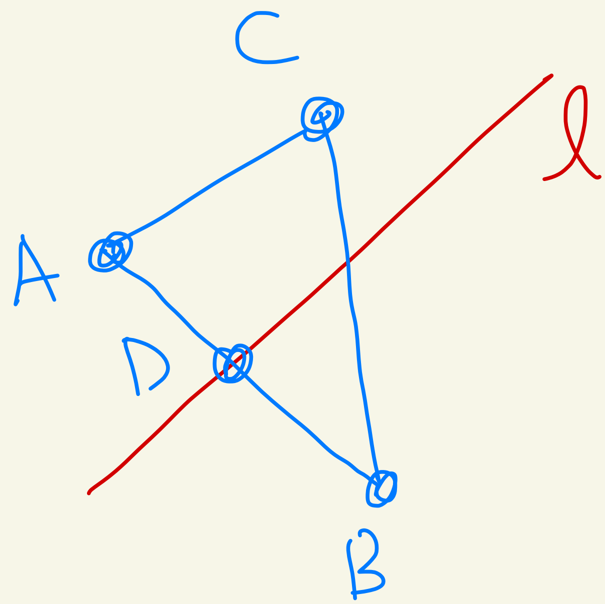
<u>Case 1:</u> Suppose $\ell \cap \overline{AC} \neq \phi$.

Then we're done.

<u>Case 2:</u> Suppose $\ell \cap \overline{AC} = \phi$.

We will show this implies $\ell \cap \overline{BC} \neq \phi$.

We know $\boxed{A \notin \ell}$ since $\ell \cap \overline{AC} = \phi$ and $A \in \overline{AC}$.



Why is $B \notin \ell$?

Suppose $B \in \ell$.

Then $D \in \ell$ and $B \in \ell$ and $D \neq B$ (because $A-D-B$) gives that $\ell = \overleftrightarrow{DB}$.

But $A \in \overleftrightarrow{DB}$ since $A-D-B$.

Contradiction since $A \notin \ell$.

Thus, $\boxed{B \notin \ell}$,

Also, $D \in \overline{AB}$ since $A - D - B$.

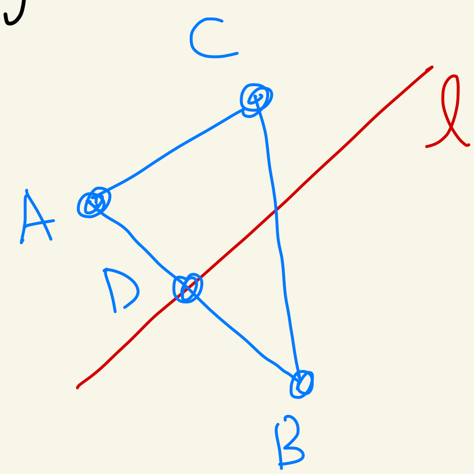
So, $\boxed{\overline{AB} \cap \ell \neq \emptyset} \leftarrow \boxed{D \in \overline{AB} \cap \ell}$

By the PSA, A and B are on opposite sides of ℓ. HW7 #6

Note that $C \notin \ell$ since $\ell \cap \overline{AC} = \emptyset$.

Since $A, C \notin \ell$ and $\ell \cap \overline{AC} = \emptyset$ we know A, C are on the same side of ℓ by PSA HW7 #6

Since A, B are on opposite sides of ℓ and A, C are on the



same side of ℓ, we know B, C are on opposite sides of ℓ.] HW7 #8

Thus, $\overline{BC} \cap \ell \neq \phi$.] HW7 #6

$\blacksquare$

Theorem: Let $(\mathcal{P}, \mathcal{L}, d)$ be a metric geometry that satisfies Pasch's Postulate (PP). Then it satisfies PSA.

pf: See notes.

$\blacksquare$

Def: If a metric geometry $(\mathscr{P}, \mathscr{L}, d)$ satisfies the PSA (or equivalently PP) then we call it a Pasch geometry.

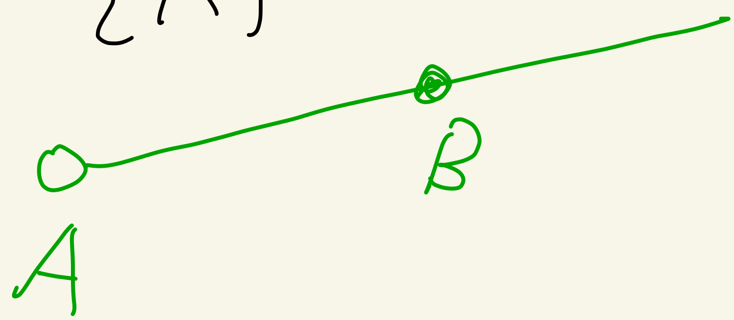
Ex: Two examples of Pasch geometries are the Euclidean and Hyperbolic planes.

Topic 9 – The Crossbar Theorem

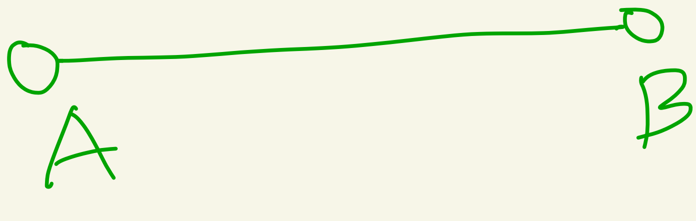
<u>Def:</u> Let $(\mathcal{P}, \mathcal{L}, d)$ be a metric geometry. Let $A, B \in \mathcal{P}$ with $A \neq B$.

The <u>interior of the ray</u> $\overrightarrow{AB}$ is

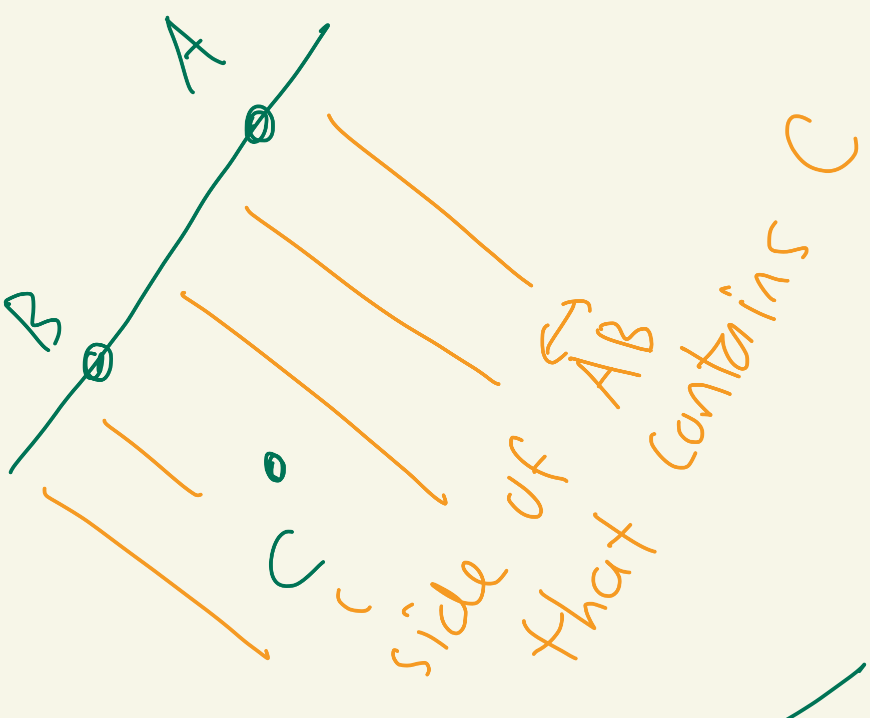
$$\text{int}(\overrightarrow{AB}) = \overrightarrow{AB} - \{A\}$$

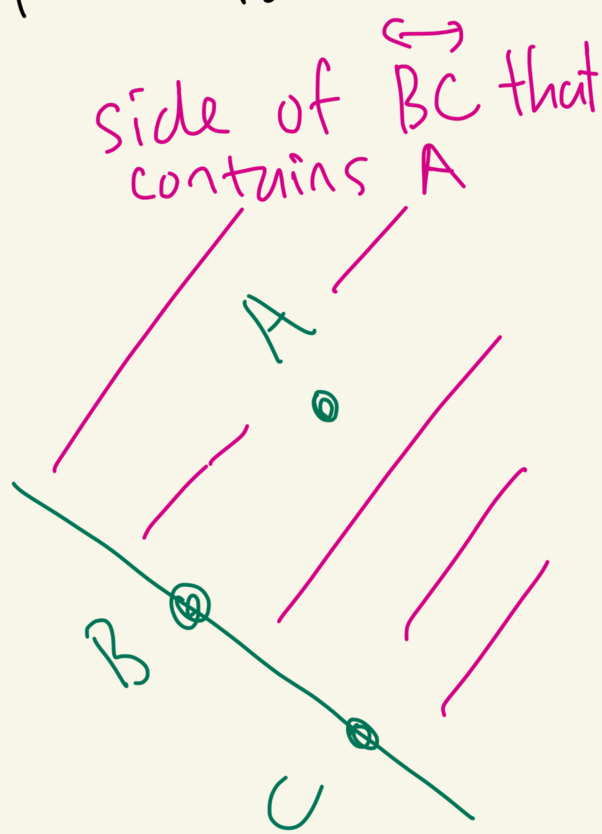


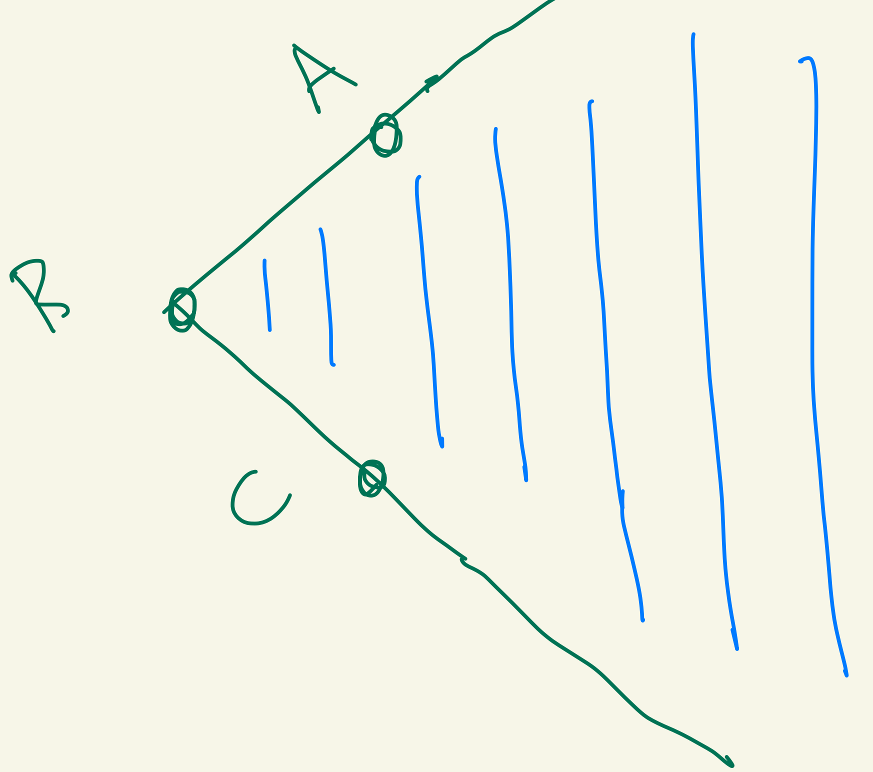
The <u>interior of the segment</u> $\overline{AB}$ is $\text{int}(\overline{AB}) = \overline{AB} - \{A, B\}$



__Def:__ Let $(\mathscr{P}, \mathscr{L}, d)$ be a Pasch geometry. Let A, B, C be noncollinear points from $\mathscr{P}$. The __interior__ of $\angle ABC$, denoted by $\text{int}(\angle ABC)$, consists of the intersection of the side of $\overleftrightarrow{AB}$ that contains C with side of $\overleftrightarrow{BC}$ that contains A.





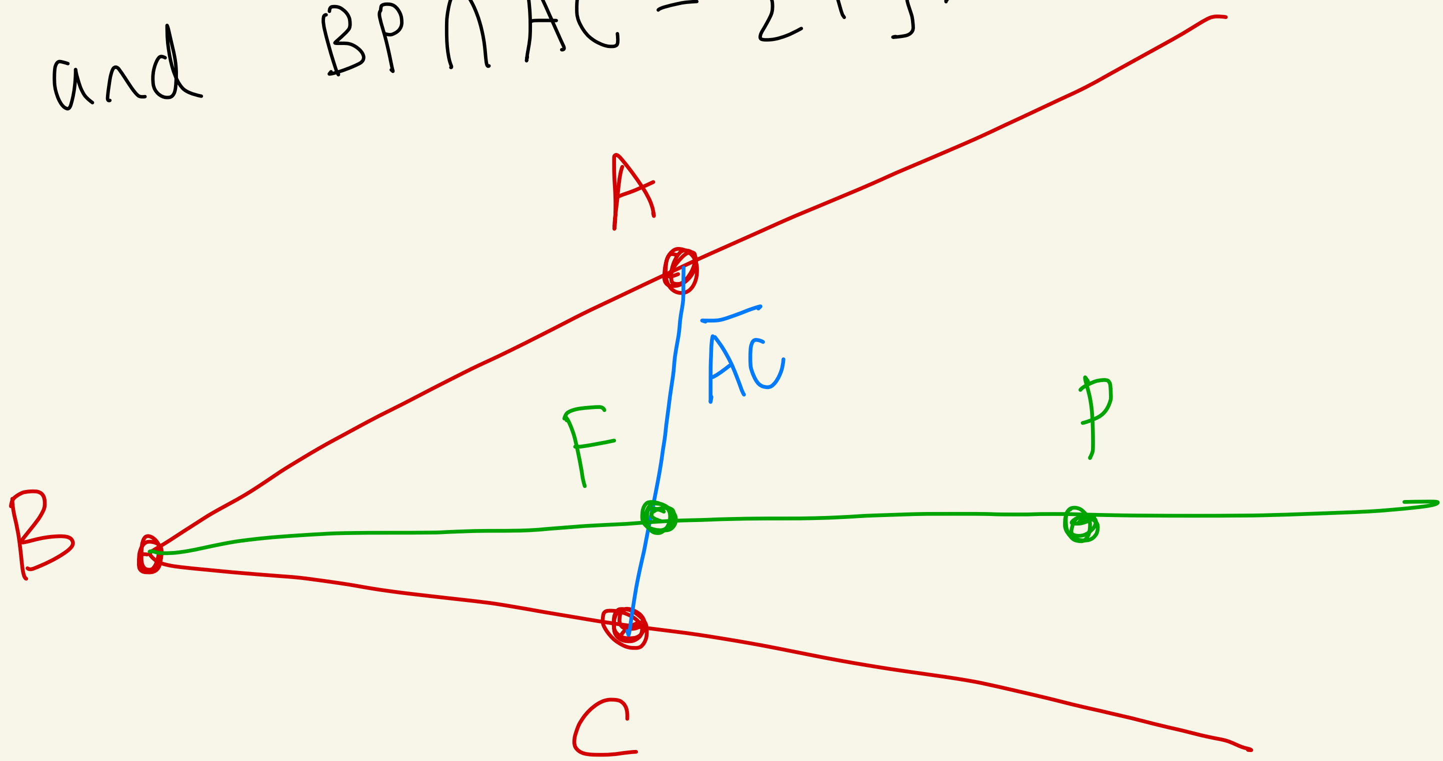


$\operatorname{int}(\angle ABC)$

Theorem (Cross-bar theorem)

Let $(\mathscr{P}, \mathscr{L}, d)$ be a Pasch geometry. Let $A, B, C, P \in \mathscr{P}$ where A, B, C are noncollinear.

If $P \in \operatorname{int}(\angle ABC)$, then there exists a unique point $F \in \mathscr{P}$ where $A - F - C$ and $\overrightarrow{BP} \cap \overline{AC} = \{F\}$.



Proof: See notes $\blacksquare$