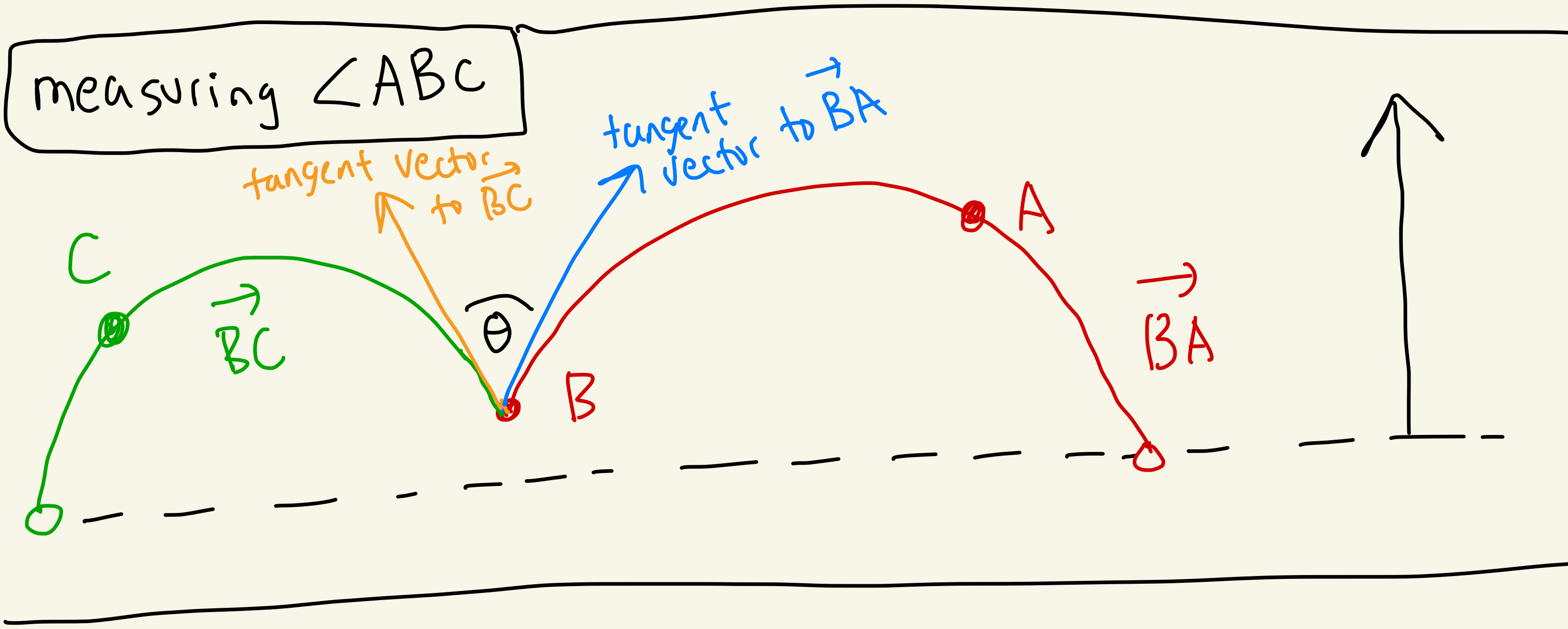
Math 4300

11/1/23

Let's now define an angle measure in the hyperbolic plane.

We need to define tangent vectors to do this.



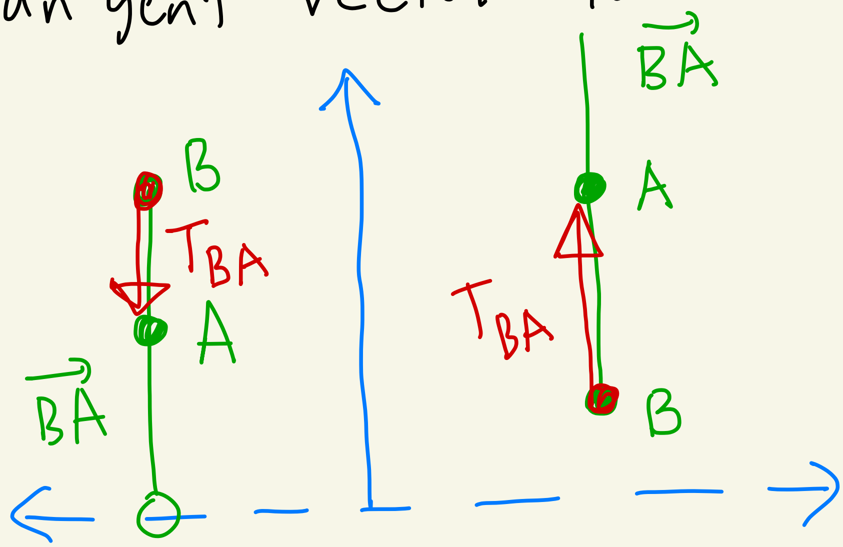
How to make the tangent vector

Let $A = (x_a, y_a)$ and $B = (x_b, y_b)$ be in the hyperbolic plane.

We want to define a tangent vector to B on the ray $\overrightarrow{BA}$.

Case 1: Suppose A and B lie on a vertical line, so $x_a = x_b$. In this case, define the tangent vector to be

$$T_{BA} = A - B$$
$$= (0, y_b - y_a)$$



Case 2: Suppose A and B both lie on $_{c}L_{r}$. Let's use calculus to motivate our def.

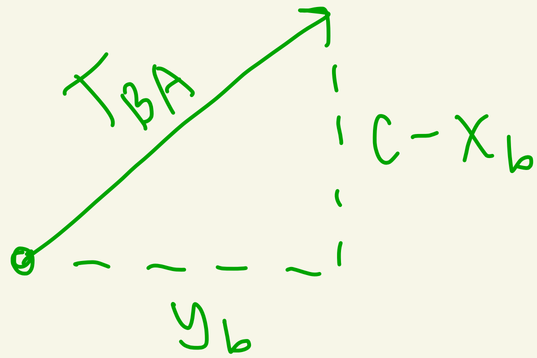
Differentiate $(x-c)^2 + y^2 = r^2$

to get $2(x-c) + 2y\frac{dy}{dx} = 0$

So, $\frac{dy}{dx} = \frac{c-x}{y}$.

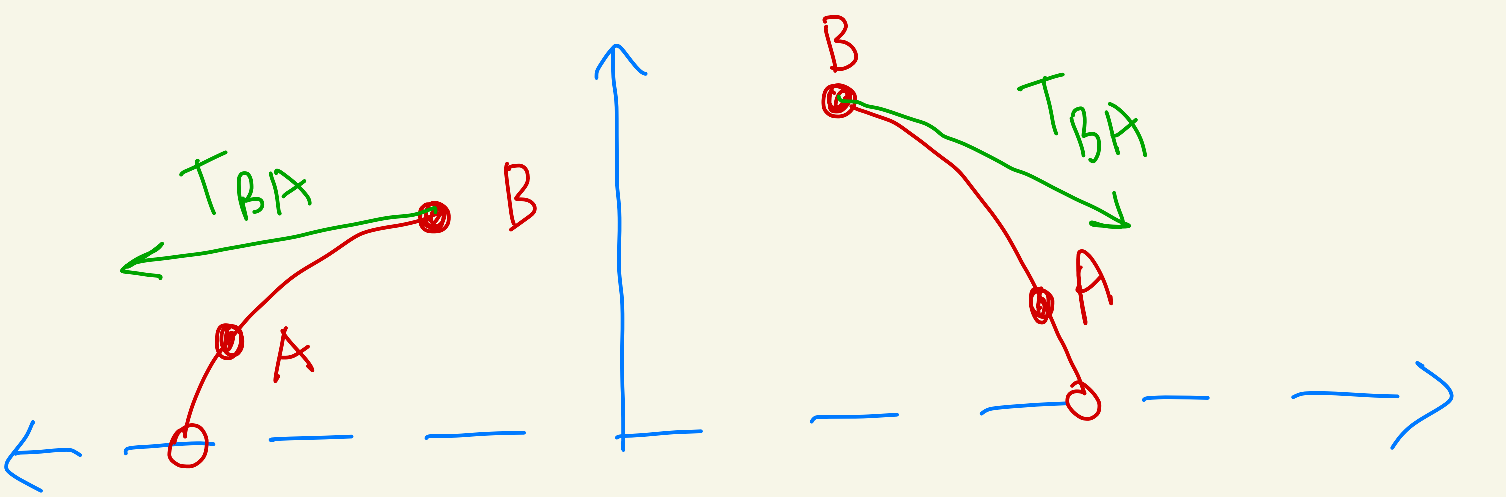
At B this would be $\boxed{\dfrac{dy}{dx} = \dfrac{c - x_b}{y_b}}$

Both $T_{BA} = \pm (y_b, c - x_b)$ both have slope $\dfrac{c-x_b}{y_b}$.



Pick a + if A is to the right of B

Pick a − if A is to the left of B.



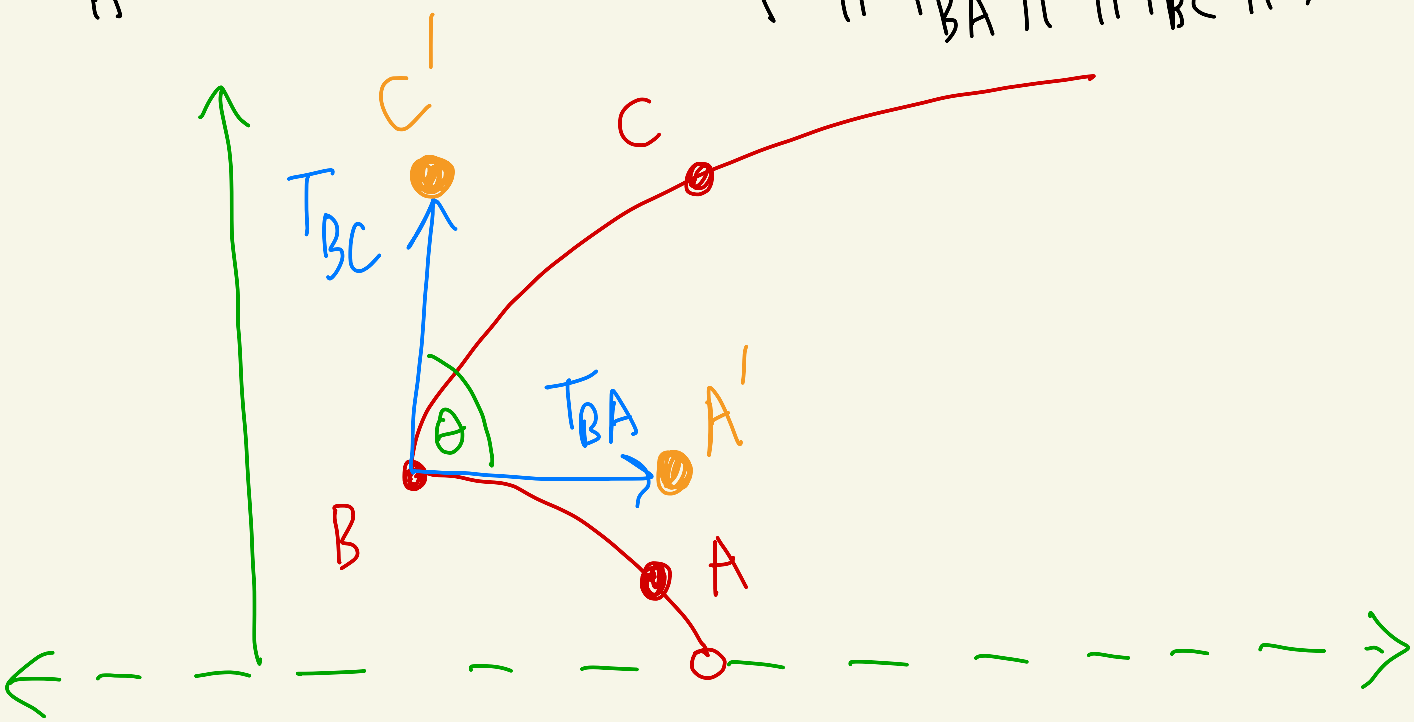
Summary: $A = (x_a, y_a)$, $B = (x_b, y_b)$

Define:

$$T_{BA} = \begin{cases} (0, y_a - y_b), & \text{if } \overleftrightarrow{AB} \text{ is a vertical line} \\ (y_b, c - x_b), & \text{if } \overleftrightarrow{AB} = {}_cL_r \text{ and } A \text{ is to the right of } B \text{ (ie } x_b < x_a) \\ -(y_b, c - x_b), & \text{if } \overleftrightarrow{AB} = {}_cL_r \text{ and } A \text{ is to the left of } B \text{ (ie } x_a < x_b) \end{cases}$$

Def: Let A, B, C be non-collinear points in the hyperbolic plane $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H, d_H)$. The hyperbolic measure m_H on $\angle ABC$ is defined as

$$m_H(\angle ABC) = \cos^{-1}\left(\frac{\langle T_{BA}, T_{BC}\rangle}{\|T_{BA}\|\,\|T_{BC}\|}\right)$$



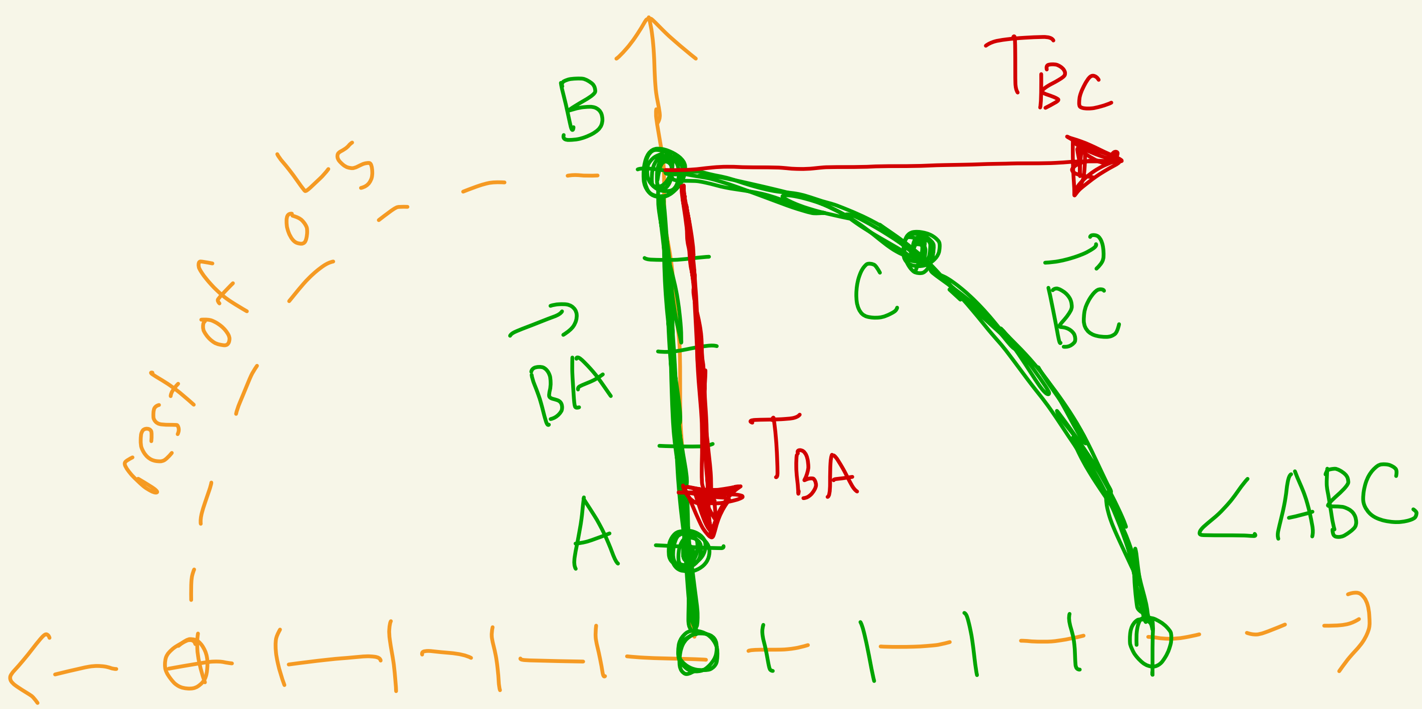
In terms of the Euclidean measure if you let $A' = B + T_{BA}$ and $C' = B + T_{BC}$ then

$$\underbrace{m_H(\angle ABC)}_{\text{hyperbolic}} = \underbrace{m_E(\angle A'BC')}_{\text{Euclidean}}$$

$\boxed{\text{Ex:}}$ In the hyperbolic plane find $m_H(\angle ABC)$ where $A=(0,1)$, $B=(0,5)$, $C=(3,4)$.

A and B lie on ${}_0L$

B and C lie on ${}_0L_5$



$$T_{BA} = A - B = (0, 1-5) = (0, -4)$$

$$T_{BC} = +(y_b, C - X_b) = (5, 0-0)$$
$$= (5, 0)$$

C is to right of B

Thus,

$$m_H(\angle ABC) = \cos^{-1}\left(\frac{\langle T_{BA}, T_{BC}\rangle}{\|T_{BA}\| \cdot \|T_{BC}\|}\right)$$

$$= \cos^{-1}\left(\frac{\langle (0,-4), (5,0)\rangle}{\|(0,-4)\| \, \|(5,0)\|}\right)$$

$$= \cos^{-1}\left(\frac{0(5) + (-4)(0)}{4 \cdot 5}\right)$$

$$= \cos^{-1}(0) = \boxed{90^\circ}$$

Thm: m_H is an angle measure and so $\mathcal{P} = (\mathbb{H}, \mathcal{L}_H, d_H, m_H)$ is a proctractor geometry.

pf: Millman/Parker 5.4. $\blacksquare$