Math 4300
8/23/23

Topic 1 - Abstract geometries and Incidence geometries

<u>Def:</u> An <u>abstract geometry</u> $(\mathcal{P}, \mathcal{L})$ consists of a non-empty set $\mathcal{P}$, whose elements are called <u>points</u>, and a non-empty set $\mathcal{L}$, whose elements are called <u>lines</u>, such that:

(i) If $\ell \in \mathcal{L}$, then $\ell \subseteq \mathcal{P}$

[a line is a set of points]

(ii) For every two points $P, Q \in \mathcal{P}$ there exists a line $\ell \in \mathcal{L}$ where $P \in \ell$ and $Q \in \ell$

[there exists a line ℓ through P and Q]

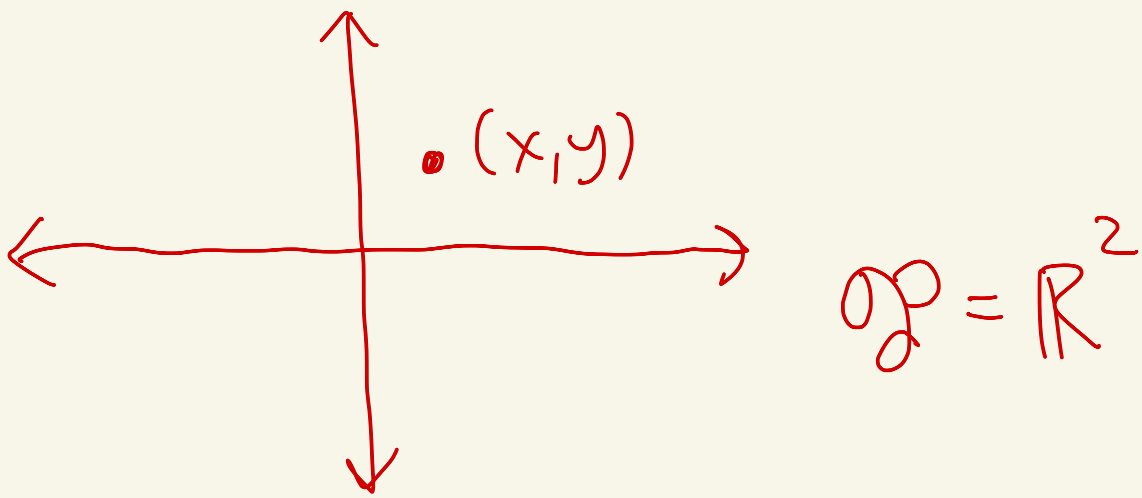
(iii) Every line $\ell \in \mathcal{L}$ contains at least two points.

<u>Note:</u> If $P \in \ell$ we say that "P lies on ℓ" or "ℓ passes through P" or "ℓ goes through P"

<u>Def:</u> Let $(\mathcal{P}, \mathcal{L})$ be an abstract geometry. Let $\ell_1, \ell_2 \in \mathcal{L}$ be lines. We say that ℓ_1 and ℓ_2 are <u>parallel</u>, written $\ell_1 \parallel \ell_2$, if either $\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2 = \emptyset$

Def: (Euclidean plane)

Let $\mathcal{P} = \mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\}$

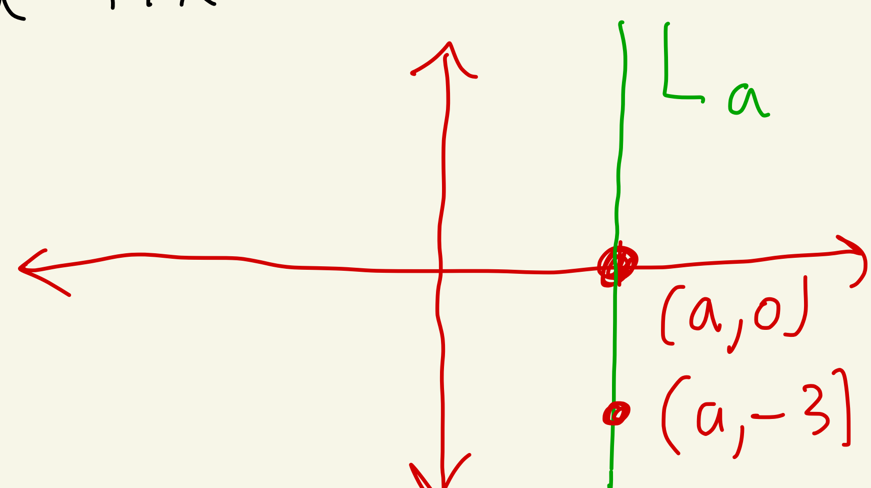


We define two kinds of lines.

A vertical line is of the form

$$L_a = \{(x,y) \in \mathbb{R}^2 \mid x = a\}$$

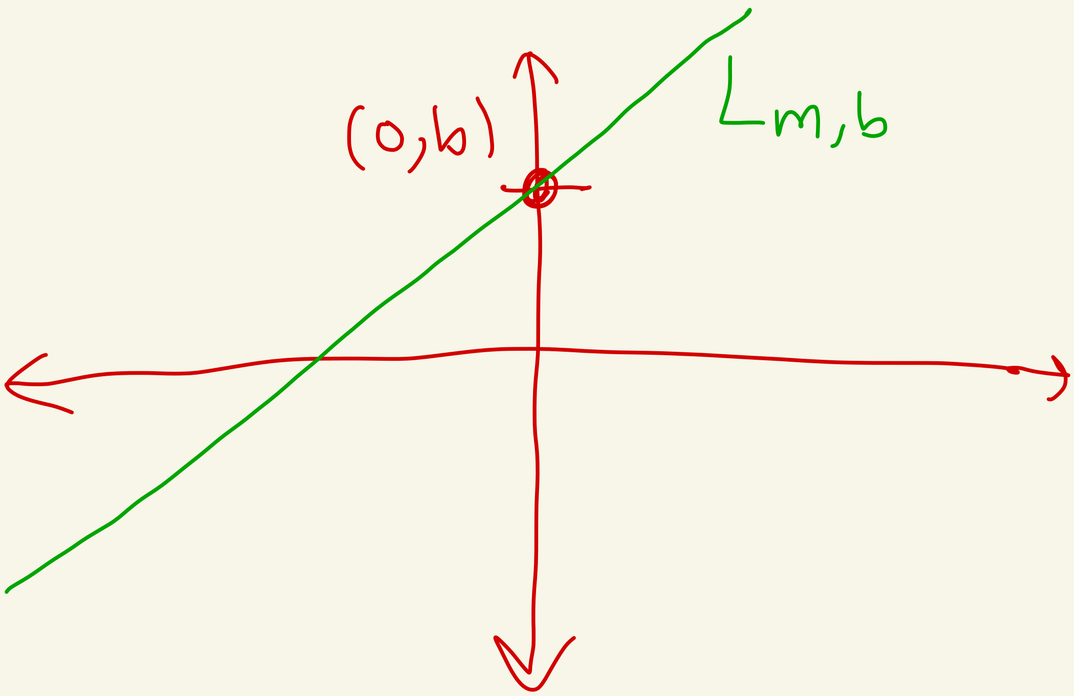
for a fixed $a \in \mathbb{R}$.



A <u>non-vertical line</u> is of the form

$$L_{m,b} = \{ (x,y) \in \mathbb{R}^2 \mid y = mx + b \}$$

where m, b are fixed real numbers.



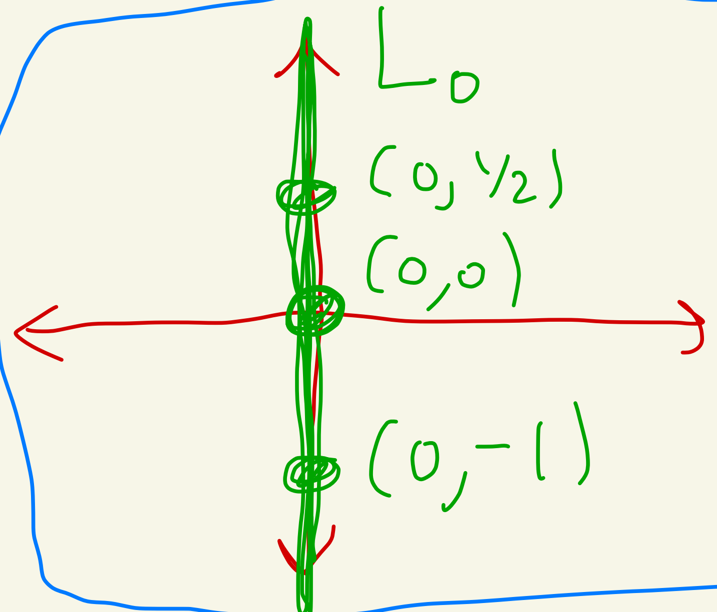
Let $\mathcal{L}_E$ consist of all vertical and non-vertical lines

Let $\mathcal{E} = (\mathbb{R}^2, \mathcal{L}_E)$

$\mathcal{E}$ is called the <u>Euclidean plane</u>

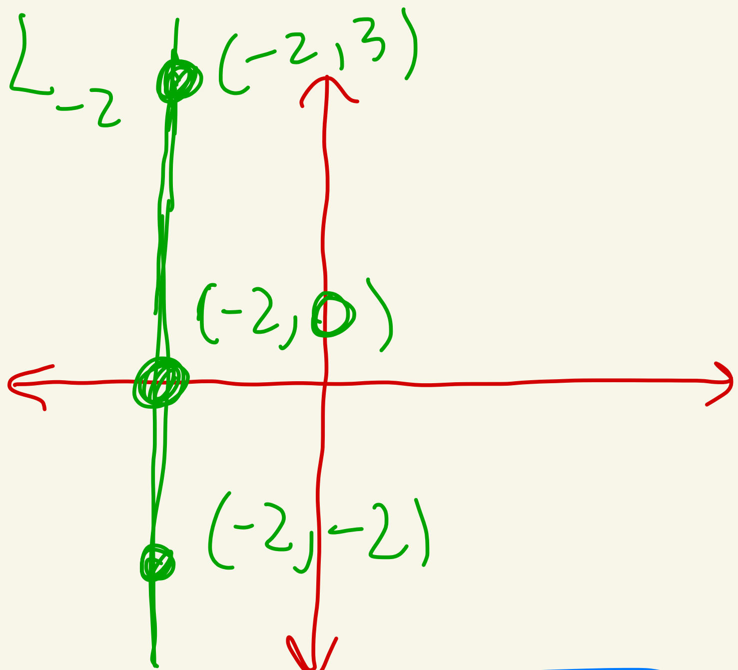
Ex: Consider the Euclidean plane $\mathcal{E} = (\mathbb{R}^2, \mathcal{L}_E)$.

Here are some vertical lines:



$(0, -\frac{1}{2}) \in L_0$

$(0, -\frac{1}{2})$ lies on L_0

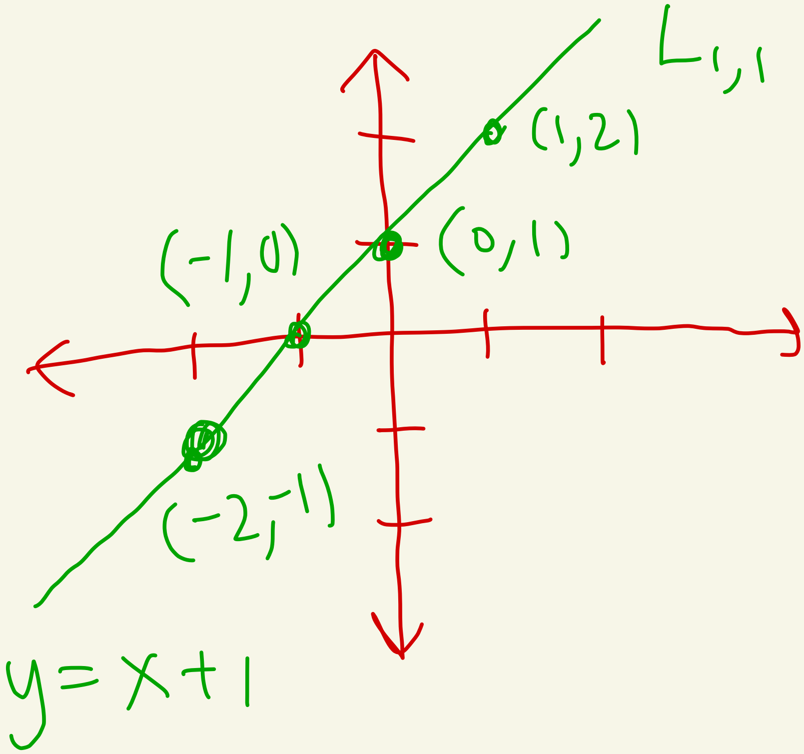


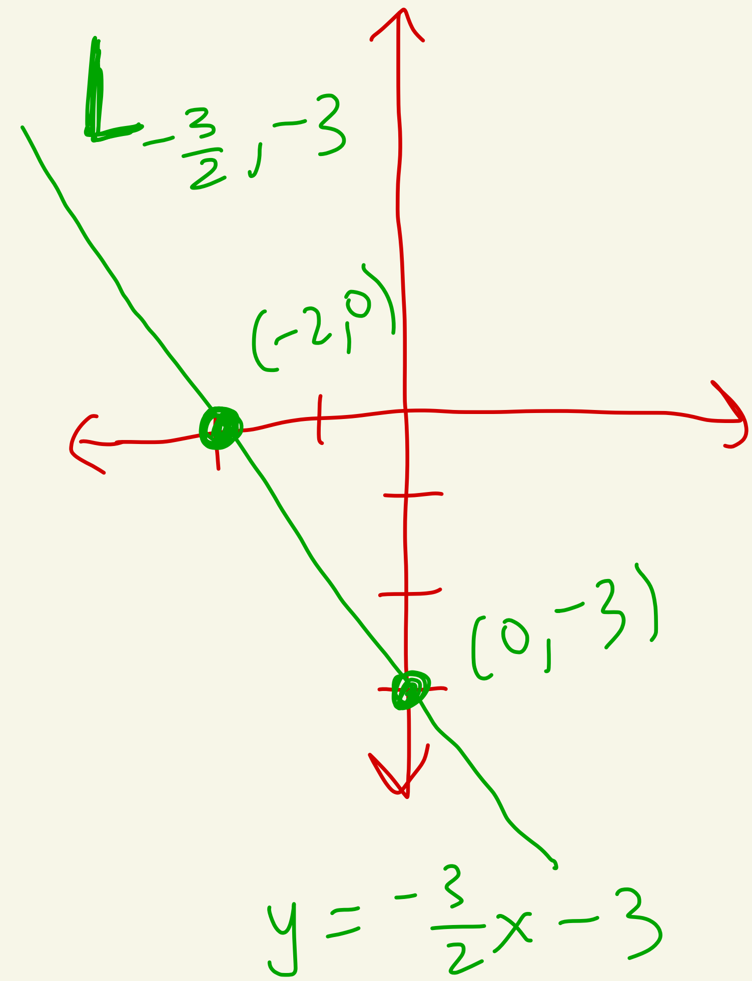
$L_0 \cap L_{-2} = \phi$

so $L_0 \parallel L_{-2}$

they are parallel

Some non-vertical lines: $\boxed{L_{m,b}}$





$L_{1,1}$ and $L_{-\frac{3}{2},-3}$ are not parallel because

$$L_{1,1} \cap L_{-\frac{3}{2},-3} = \left\{\left(-\frac{8}{5}, \frac{-3}{5}\right)\right\}$$

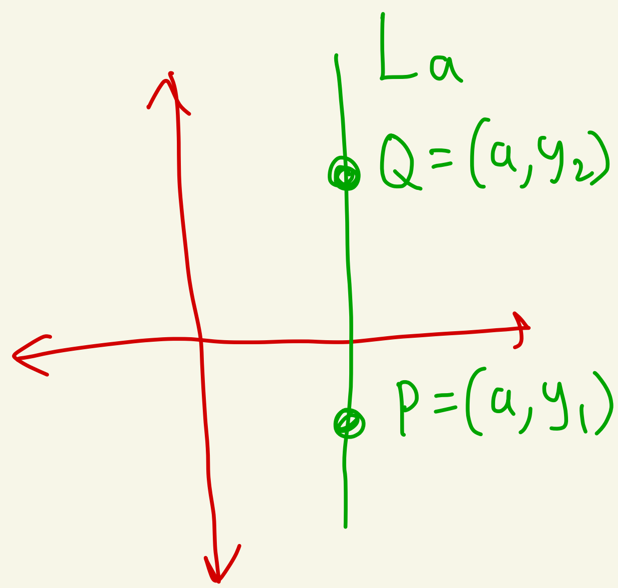
Theorem: The Euclidean plane $\mathcal{E} = (\mathbb{R}^2, \mathcal{L}_E)$ is an abstract geometry.

Proof:

$\mathbb{R}^2$ and $\mathcal{L}_E$ are both non-empty.

(i) Both $L_a \subseteq \mathbb{R}^2$ and $L_{m,b} \subseteq \mathbb{R}^2$.

(ii) Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be points from $\mathbb{R}^2$. We need to show there exists a line through them.



Case 1: Suppose $x_1 = x_2$.

Set $a = x_1 = x_2$

Then both P and Q lie on L_a.

Case 2: Suppose $x_1 \neq x_2$.

Set $\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$ ← this is defined because $x_2 - x_1 \neq 0$

Then set $\boxed{b = y_1 - m x_1}$.

We claim that both $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ lie on $L_{m,b}$.

Since $b = y_1 - m x_1$ we know $y_1 = m x_1 + b$ and so $p = (x_1, y_1)$ lies on $L_{m,b}$.

What about $Q = (x_2, y_2)$?

We have that

$$y_2 - m x_2 - b = y_2 - m x_2 - (y_1 - m x_1)$$

$$= (y_2 - y_1) - m(x_2 - x_1)$$

$$= (y_2 - y_1) - \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x_2 - x_1)$$

$$= (y_2 - y_1) - (y_2 - y_1) = 0.$$

So, $Q = (x_2, y_2)$ also lies on $L_{m,b}$.

(iii) A vertical line L_a $\leftarrow$ $x=a$

contains at least two points,
for example $(a,0)$ and $(a,1)$.

A non-vertical line $L_{m,b}$
contains at least two points,
for example $(1, m+b)$
and $(2, 2m+b)$ are
both on $L_{m,b}$. $\leftarrow$ $y = mx+b$

$\boxed{}$