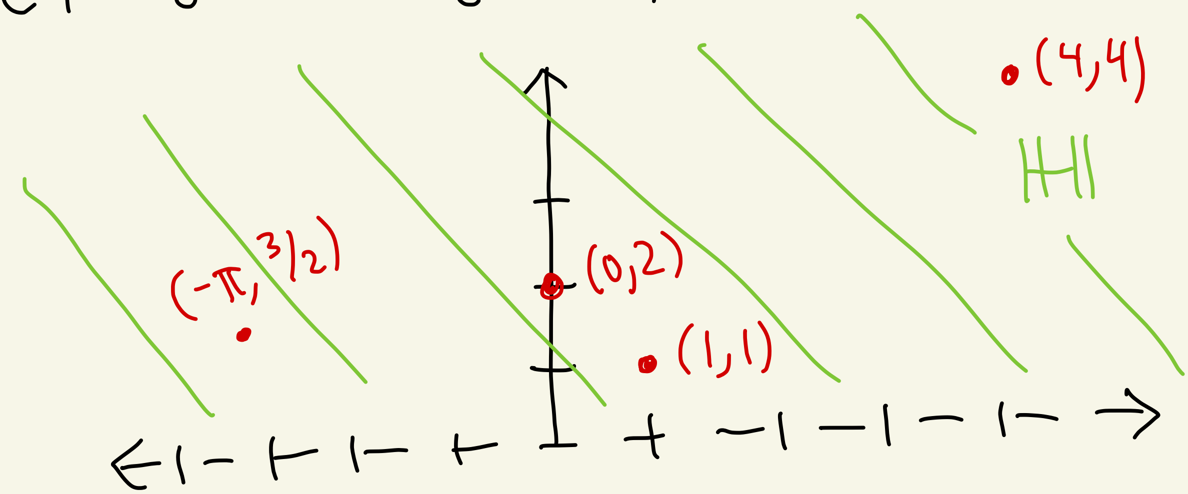
Math 4300
8/28/23

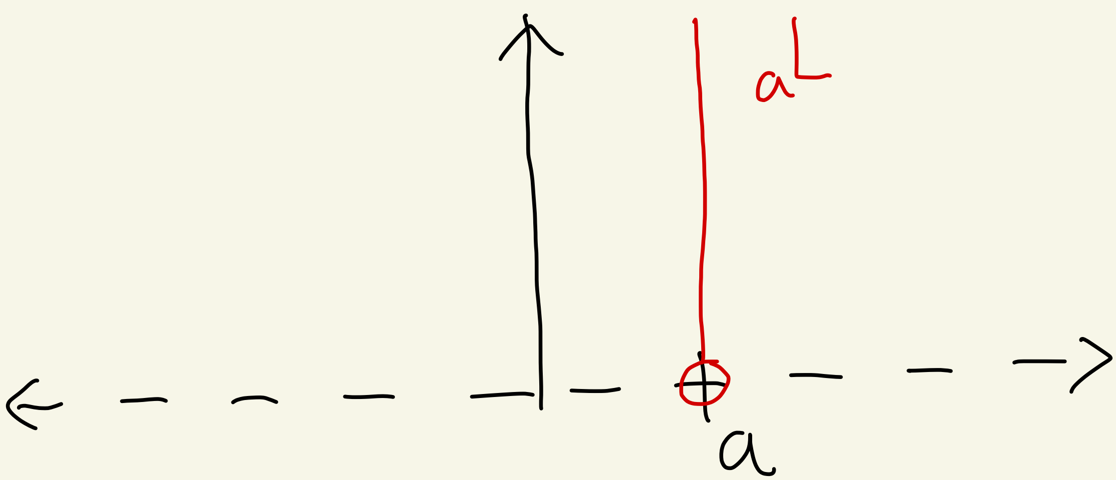
## Def:

Let $\mathcal{P} = \mathbb{H} = \{(x,y) \mid x,y \in \mathbb{R} \text{ and } y > 0\}$.




A <u>type I line</u> is of the form

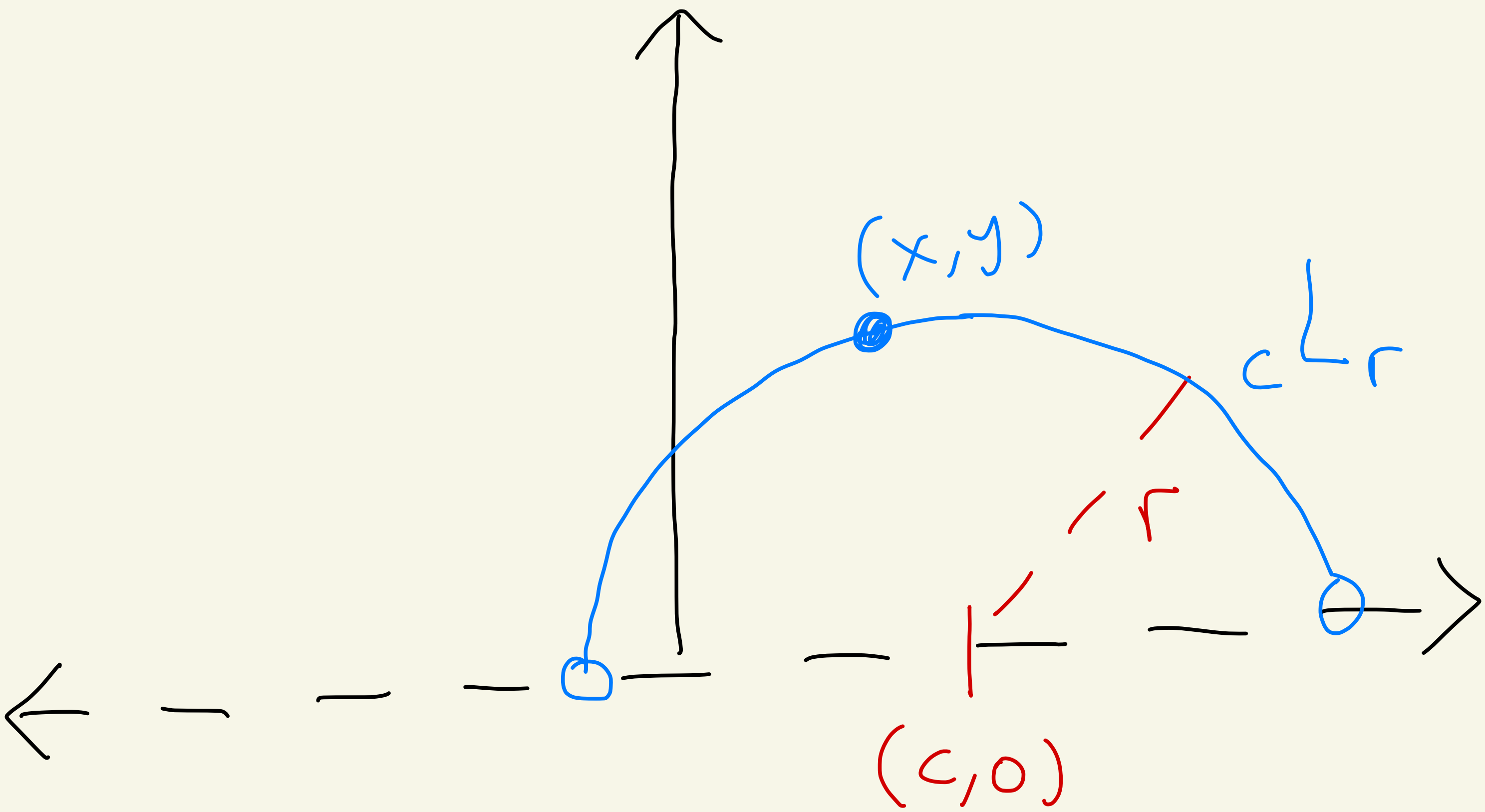
$${}_aL = \{(x,y) \in \mathbb{H} \mid x = a\}$$



A <u>type II line</u> is of the form

$${}_cL_r = \{(x,y) \in \mathbb{H} \mid (x-c)^2 + y^2 = r^2\}$$

where $c \in \mathbb{R}$, $r \in \mathbb{R}$, $r > 0$.

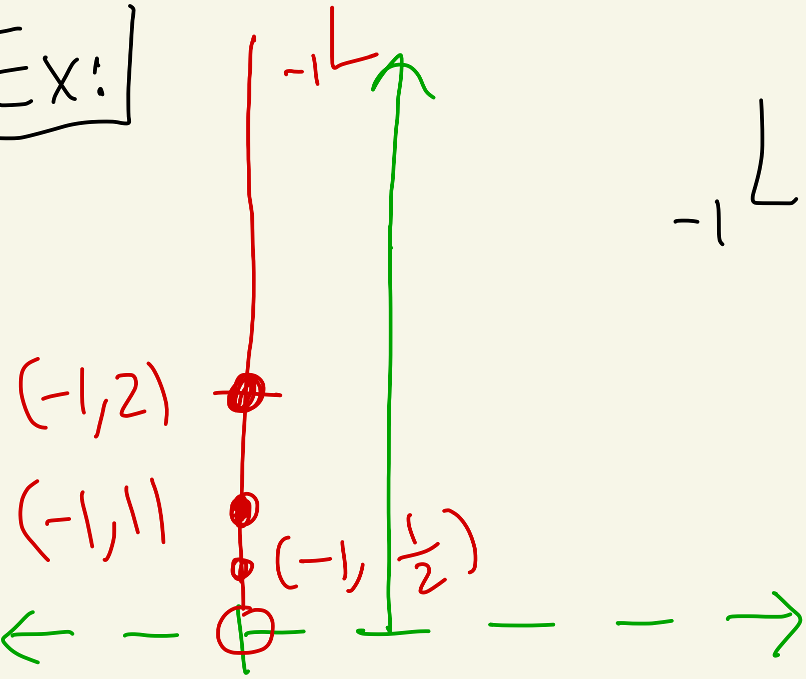



Let $\mathcal{L}_H$ be the set consisting of all type I and type II lines

Let $\mathcal{H} = (\underbrace{\mathbb{H}}_{\text{points}}, \underbrace{\mathcal{L}_H}_{\text{lines}})$.

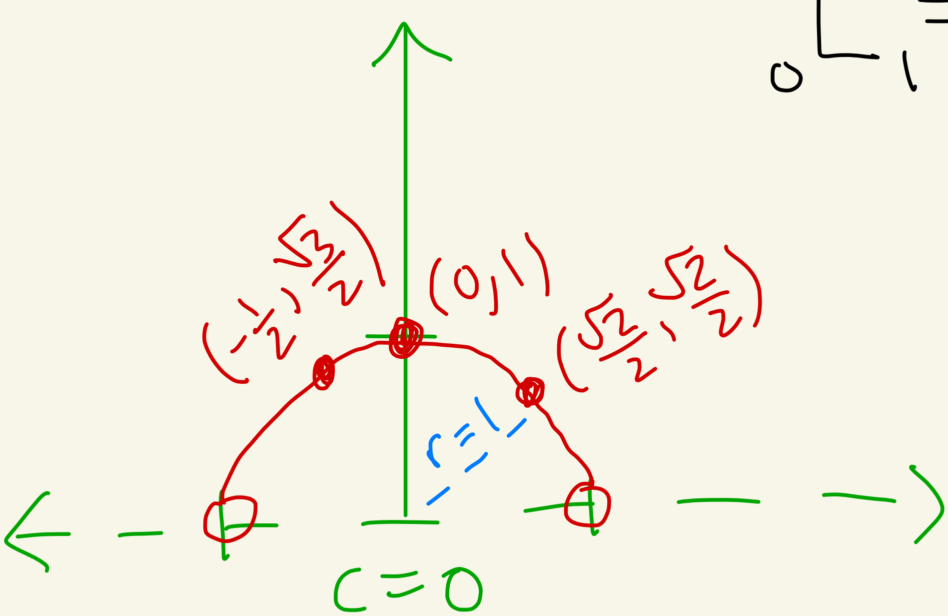
$\mathcal{H}$ is called the *Hyperbolic plane* or the *Poincare upper-half plane*.

Ex:

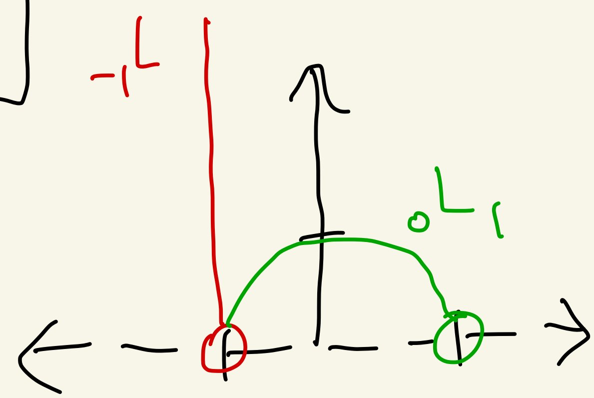



$$_{-1}L = \{(x,y)\in \mathbb{H} \mid x=-1\}$$

$$_{0}L_{1} = \{(x,y)\in \mathbb{H} \mid (x-0)^2+y^2=1^2\}$$




Note that $_{-1}L \parallel {}_{0}L_{1}$.




Why?

Suppose $(x,y)\in {}_{-1}L \cap {}_{0}L_{1}$

Since $(x,y)\in {}_{-1}L$ we know $x=-1$

Plugging $(x,y)=(-1,y)$ into

$$(x-0)^2+y^2=1^2 \leftarrow \text{eqn for } {}_0L_1$$

You get

$$(-1)^2+y^2=1^2$$

or $y=0$.

But then $(x,y)=(-1,0)$ which is not in $\mathbb{H}$.

Thus, ${}_{-1}L \cap {}_0L_1 = \emptyset$ $\blacksquare$

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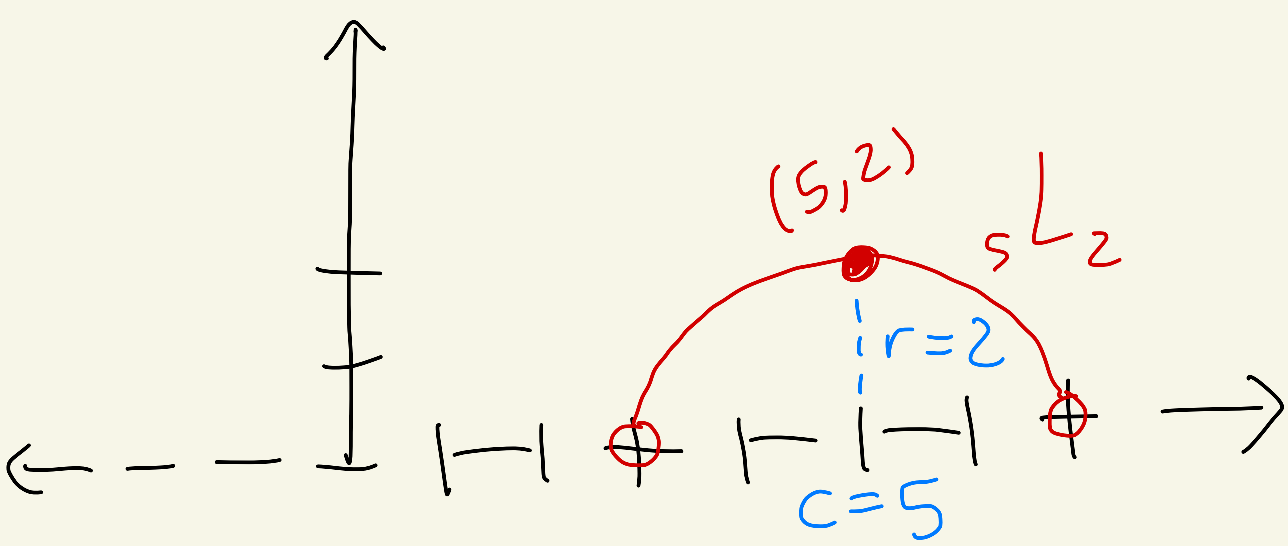
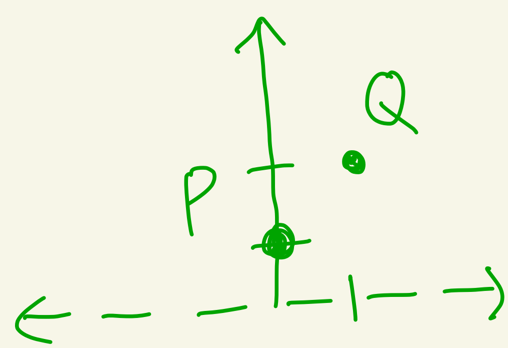
Ex:

$${}_5L_2=\{(x,y)\in\mathbb{H} \mid (x-5)^2+y^2=2^2\}$$



HW problem: Show ${}_5L_2$ and ${}_0L_1$ are parallel

Ex: In the hyperbolic plane find a line that goes through $P=(0,1)$ and $Q=(1,2)$.




Since P and Q have different x-coordinates they don't both lie on a type I line. Is there a type II line that they lie on?

Plug $P=(0,1)$ and $Q=(1,2)$ into the equation: $(x-c)^2+y^2=r^2$.

$$\boxed{\begin{aligned}(0-c)^2+1^2&=r^2 \quad (1) \leftarrow \text{plug P in}\\ (1-c)^2+2^2&=r^2 \quad (2) \leftarrow \text{Plug Q in}\end{aligned}}$$

This gives:

$$\boxed{\begin{aligned} c^2+1 &= r^2 & (1)\\ c^2-2c+5 &= r^2 & (2)\end{aligned}}$$

Subtract (1) − (2) to get:

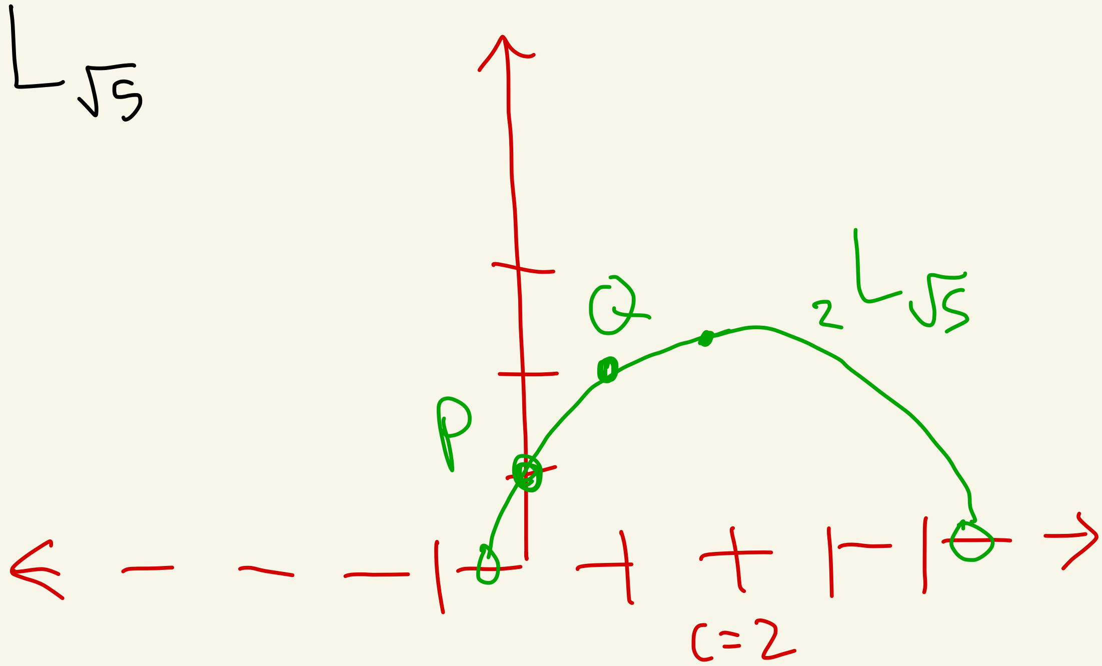
$$2c-4=0$$

That gives $\boxed{c=2.}$

Plug $c=2$ into (1) to get $r^2 = 2^2+1=5$

So, $\boxed{r=\sqrt{5}}$

Thus, $P=(0,1)$ and $Q=(1,2)$ both lie on ${}_2L_{\sqrt{5}}$




$\sqrt{5} \approx 2.236$

<u>Theorem:</u> The hyperbolic plane $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H)$ is an abstract geometry.

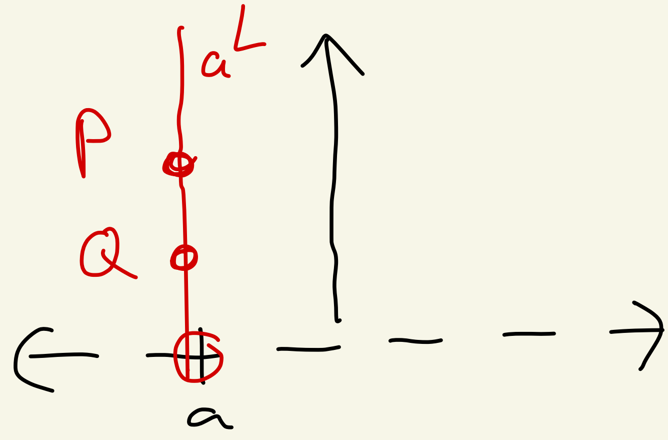
<u>proof:</u> By def $\mathbb{H} \neq \phi$, $\mathcal{L}_H \neq \phi$.

(i) If $l \in \mathcal{L}_H$, then $l \subseteq \mathbb{H}$.

(ii) Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be in $\mathbb{H}$. We must find a line that goes through them.

<u>case 1:</u> Suppose $x_1 = x_2 = a$.

Then, $P = (a, y_1)$, $Q = (a, y_2)$ lie on ${}_aL$.



Case 2: Suppose $x_1 \neq x_2$.

Then, P, Q don't both lie on a type I line. What about a type II line?

We must solve:

$$\boxed{\begin{aligned}(x_1-c)^2+y_1^2&=r^2 \quad (1)\\ (x_2-c)^2+y_2^2&=r^2 \quad (2)\end{aligned}}$$

plug P and Q into $(x-c)^2+y^2=r^2$

This becomes

$$\boxed{\begin{aligned}x_1^2-2cx_1+c^2+y_1^2&=r^2 \quad (1)\\ x_2^2-2cx_2+c^2+y_2^2&=r^2 \quad (2)\end{aligned}}$$

Subtract (1) − (2) to get

$$\boxed{x_1^2-2cx_1+y_1^2-x_2^2+2cx_2-y_2^2=0}$$

Then

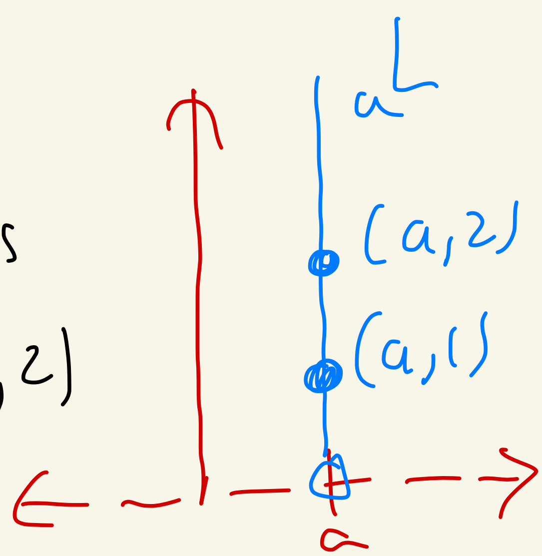
$$\boxed{c = \frac{y_2^2 - y_1^2 + x_2^2 - x_1^2}{2(x_2 - x_1)}}$$

Set $\boxed{r = \sqrt{(x_1 - c)^2 + y_1^2}}$ ← where $c$ is

In HW you verify that $P$ and $Q$ indeed lie on ${}_cL_r$ where $c, r$ are defined above.

(iii) We need to show that any line has at least two points.

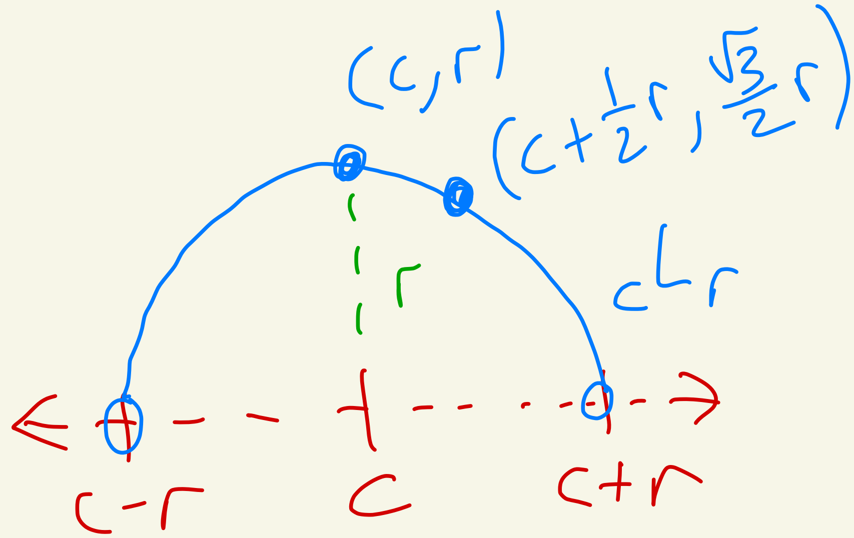
A type I line ${}_aL$ has at least $(a,1)$ and $(a,2)$



A type II line $_cL_r$ has at least the points

$(c,r)$ and

$\left(c+\frac{1}{2}r, \frac{\sqrt{3}}{2}r\right)$




check:

$$\underbrace{\left(\left(c+\frac{1}{2}r\right)-c\right)^2+\left(\frac{\sqrt{3}}{2}r\right)^2}_{(x-c)^2+y^2}=\frac{1}{4}r^2+\frac{3}{4}r^2$$

$$=r^2$$

∎