Math 4300

9/11/23

Def: Let $(\mathcal{P}, \mathcal{L})$ be an incidence geometry.
Let d be a metric on $\mathcal{P}$.
If every line $\ell \in \mathcal{L}$ has a ruler with respect to d, then we say that $(\mathcal{P}, \mathcal{L}, d)$ is a **metric geometry**

Theorem: $\mathcal{E} = (\mathbb{R}^2, \mathcal{L}_E, d_E)$ is a metric geometry. One possible set of rulers is:

$f : L_a \to \mathbb{R}$ given by $f(a, y) = y$

$f : L_{m,b} \to \mathbb{R}$ given by $f(x, mx+b) = x\sqrt{1+m^2}$

We call these the <u>standard rulers</u> for the Euclidean plane $\mathcal{E}$

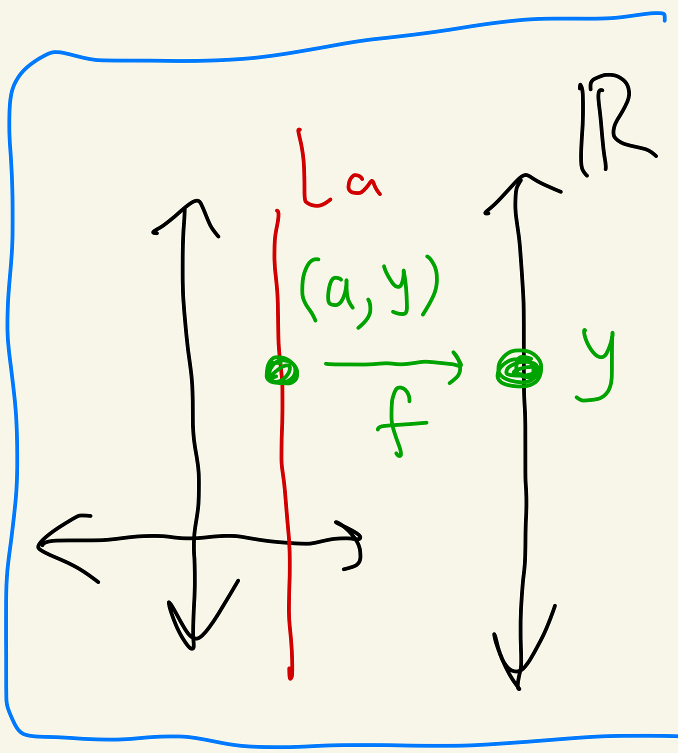
<u>proof:</u> We already proved that d_E is a distance function. We have to show that the above are rulers.

<u>case 1:</u> Let's show $f : L_a \to \mathbb{R}$ given by $f(a,y) = y$ is a ruler.

Why is f onto?

Let $y \in \mathbb{R}$

Then $(a,y) \in L_a$ and $f(a,y) = y$



What about the ruler formula?

Let $P=(a,y_1)$ and $Q=(a,y_2)$ be on L_a.

Then,

$$d_E(P,Q)=\sqrt{(a-a)^2+(y_1-y_2)^2}$$
$$=\sqrt{(y_1-y_2)^2}$$
$$=|y_1-y_2|$$
$$=|f(a,y_1)-f(a,y_2)|$$
$$=|f(P)-f(Q)|$$

By the lemma, f is a ruler.

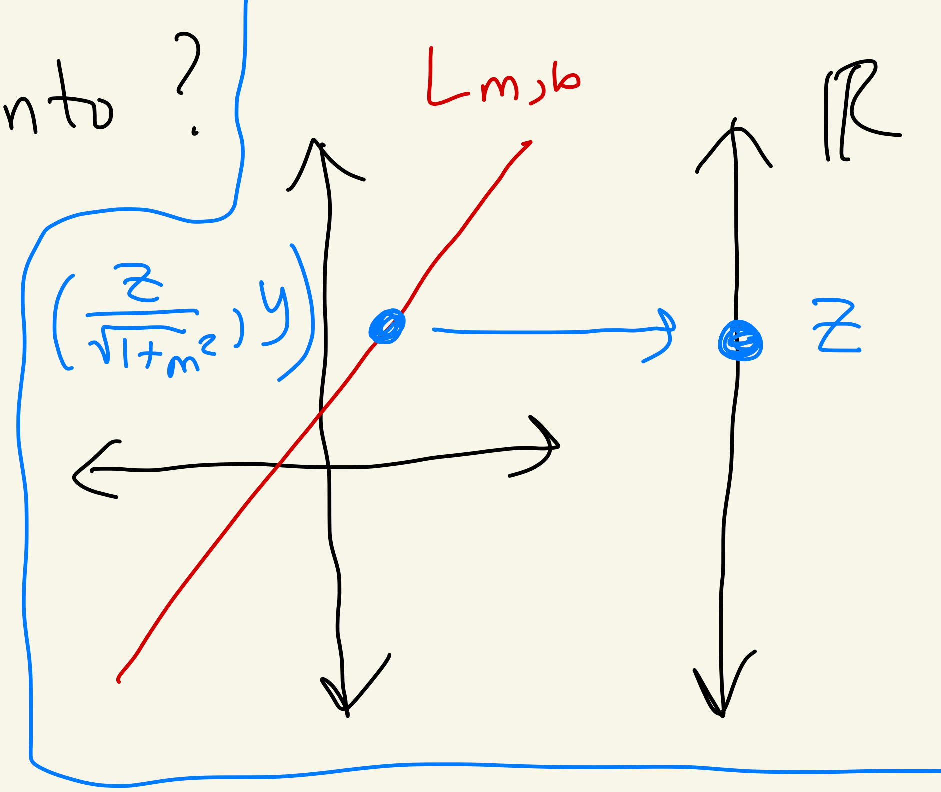
<u>Case 2:</u> Now consider the function $f: L_{m,b} \longrightarrow \mathbb{R}$ given by $f(x,mx+b)=x\sqrt{1+m^2}$

Why is f onto?

Let $z \in \mathbb{R}$

Let $\boxed{x = \frac{z}{\sqrt{1+m^2}}}$

and $y = mx+b$.



Then $f\left(\frac{z}{\sqrt{1+m^2}}, m\frac{z}{\sqrt{1+m^2}}+b\right)$

$$= \frac{z}{\sqrt{1+m^2}} \cdot \sqrt{1+m^2} = z$$

So, f is onto.

Now let's show the ruler eqn.

Let $P = (x_1, mx_1+b)$,

$Q = (x_2, mx_2+b)$ be on $L_{m,b}$.

Then,

$$d_E(P,Q) = \sqrt{(x_1-x_2)^2 + (mx_1+b-mx_2-b)^2}$$

$$= \sqrt{(x_1-x_2)^2 + (mx_1-mx_2)^2}$$

$$= \sqrt{(x_1-x_2)^2 + m^2(x_1-x_2)^2}$$

$$= \sqrt{(1+m^2)(x_1-x_2)^2}$$

$$= \sqrt{1+m^2}\,\sqrt{(x_1-x_2)^2}$$

$$= \sqrt{1+m^2} \cdot |x_1-x_2|$$

$$|a| = \sqrt{a^2}$$

$$= \left|\sqrt{1+m^2} \cdot x_1 - \sqrt{1+m^2} \cdot x_2\right|$$

$$c \cdot |a| = |ca|$$

$$c > 0$$

$$= |f(P) - f(Q)|$$

So by the lemma f is a ruler.

$\blacksquare$

For the Euclidean plane first we made a distance function and then we made rulers that respected the distance function.

For the Hyperbolic plane we reverse this and make the rulers first and then the distance function.

First we need the hyperbolic functions.

<u>Def:</u> Let $t \in \mathbb{R}$ define

$$\sinh(t) = \frac{e^{t} - e^{-t}}{2}$$

$$\cosh(t) = \frac{e^{t} + e^{-t}}{2}$$

$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} = \frac{e^{t} - e^{-t}}{e^{t} + e^{-t}}$$

$$\operatorname{sech}(t) = \frac{1}{\cosh(t)} = \frac{2}{e^{t} + e^{-t}}$$

Lemma: For any $t \in \mathbb{R}$, we have that:

(i) $[\cosh(t)]^2 - [\sinh(t)]^2 = 1$

(ii) $[\tanh(t)]^2 + [\operatorname{sech}(t)]^2 = 1$

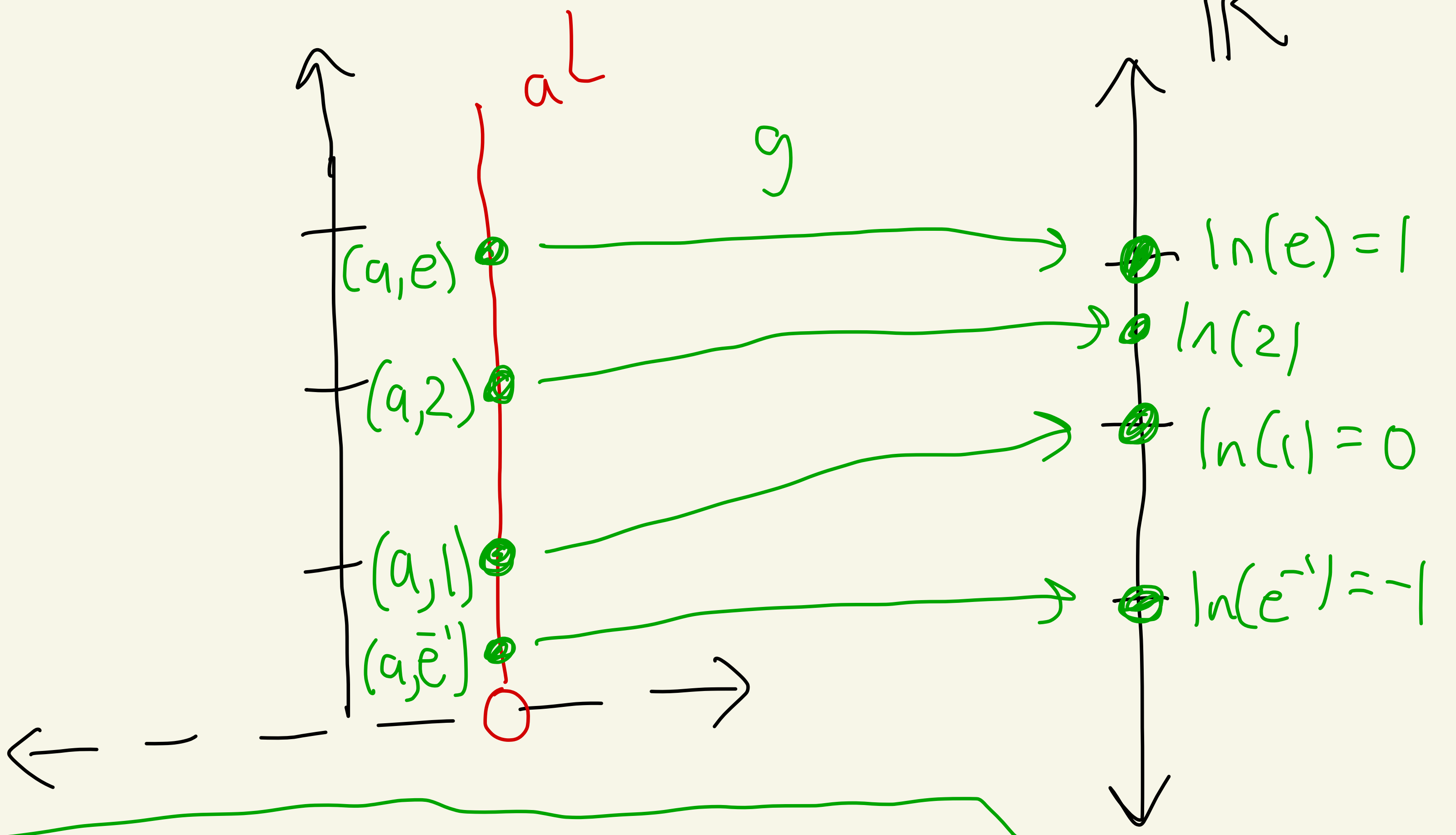
(iii) $\sec(t) > 0$

proof: See HW 2 $\blacksquare$

Let's now make bijections from the lines in the hyperbolic plane and $\mathbb{R}$. These will be our rulers.

Theorem: Consider the hyperbolic plane $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H)$.

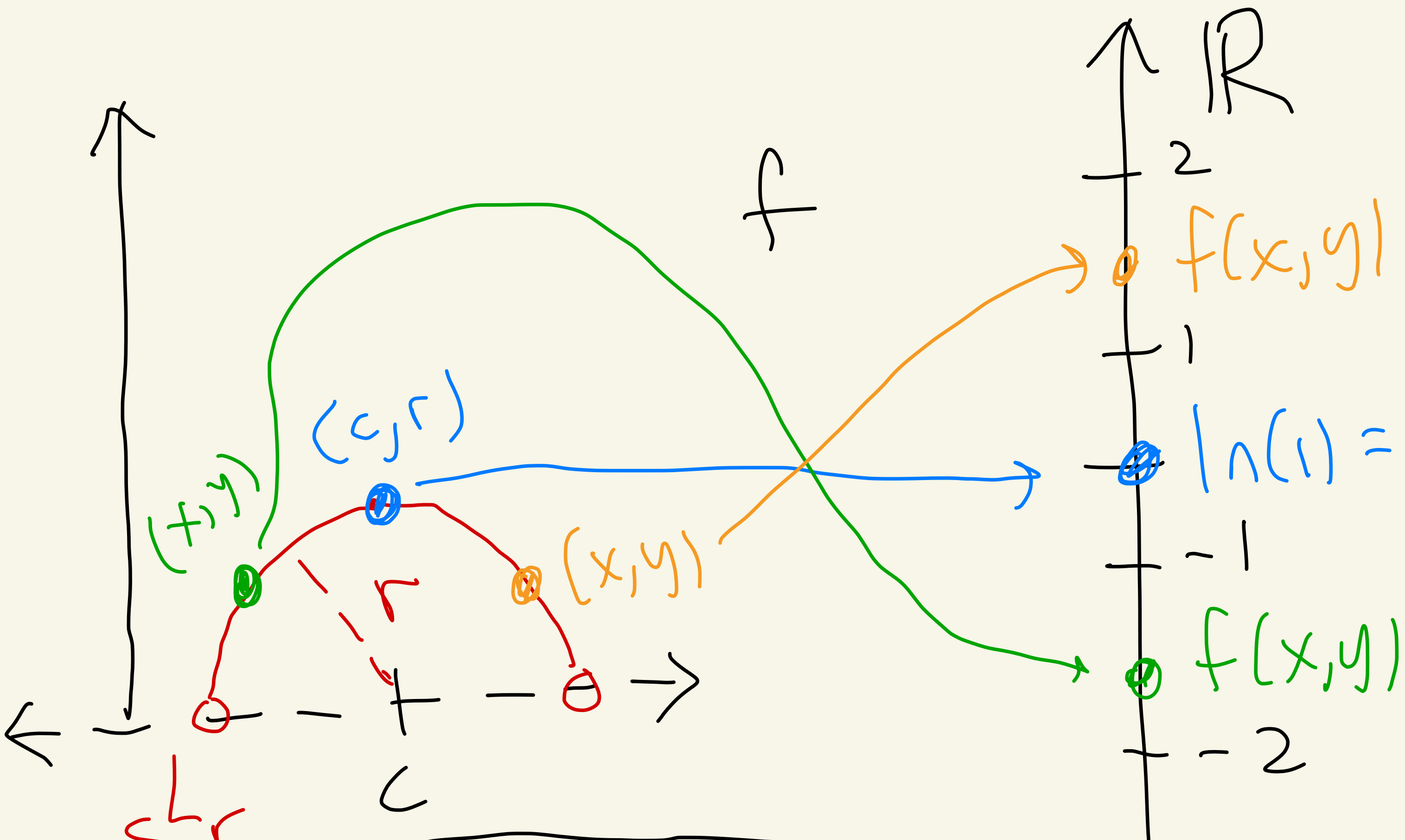
(i) The function $g : {}_aL \to \mathbb{R}$ given by $g(a,y) = \ln(y)$ is a bijection with inverse function $g^{-1}(t) = e^t$



$g(a,1) = \ln(1) = 0, \quad g(a,e^{-1}) = \ln(e^{-1}) = -1$

(ii) the function $f : {}_cL_r \to \mathbb{R}$ given by $f(x,y) = \ln\left(\frac{x-c+r}{y}\right)$ is a bijection with inverse function

$$f^{-1}(t) = \left(c + r \cdot \tanh(t),\ r \cdot \operatorname{sech}(t)\right)$$



$$\boxed{f(x,y) = \ln\left(\frac{x-c+r}{y}\right)}$$

Why is the picture as above?

$\boxed{\text{Case 1:}}$ Let $(x,y) \in {}_cL_r$ with $x < c$

We know $y < r$.

Thus, $\underbrace{y}_{>0}\underbrace{(r-y)}_{>0} > 0$

So, $yr > y^2$.

Thus, $2yr > 2y^2$.

So, $r^2 - 2ry + y^2 < \underbrace{r^2 - y^2}_{(x-c)^2 = r^2 - y^2}$

Thus, $r^2 - 2ry + y^2 < (x-c)^2$

Hence, $(r-y)^2 < (x-c)^2$

So, $(r-y)^2 < (c-x)^2$

$r - y > 0$
$c - x > 0$

Thus, $r - y < c - x$.

Hence, $x - c + r < y$.

So, $\dfrac{x-c+r}{y} < 1$

Thus, in this case $f(x,y) = \ln\left(\dfrac{x-c+r}{y}\right) < 0$.

$\boxed{\text{Case 2:}}$ Let $(x,y) \in {}_cL_r$ with $x>c$

Then $x-c>0$.

So, $x-c+r>r$.

Note $r \geq y$.

So, $x-c+r>r\geq y$

Then, $\dfrac{x-c+r}{y}>1$

So, $\ln\left(\dfrac{x-c+r}{y}\right)>0$.

So, $f(x,y)>0$ in $\mathbb{R}$.