Math 4460

1/23/23

Assumptions for the class

We will assume that the set of integers

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots\}$$

exists.

We will also assume the basic algebraic properties of $\mathbb{Z}$.

For example, if $a, b, c \in \mathbb{Z}$ we will assume:

- $a + b \in \mathbb{Z}$ (closed under $+$)
- $ab \in \mathbb{Z}$ (closed under $\cdot$)
- $a + (b + c) = (a + b) + c$
- $a(bc) = (ab)c$
- $a + b = b + a$
- $ab = ba$
- $0 + a = a + 0 = a$
- $1 \cdot a = a \cdot 1 = a$
- $a + (-a) = (-a) + a = 0$
- $a[b + c] = ab + ac$
- $[b + c]a = ba + ca$

ring axioms

We will assume all the other usual basic algebra/arithmetic facts like

- If $a > b$ then $-a < -b$.

HW 1 TOPIC – Division and Primes

Def: Let a and b be integers with $a \neq 0$. We say that a divides b if there exists an integer k where $b = ak$.
If a divides b then we say that a is a divisor of b and we write $a \mid b$.

read "a divides b"

If a does not divide b we say that a is not a divisor of b and we write $a \nmid b$.

read "a does not divide b"

Ex:

Divisors of 12: $1, 2, 3, 4, 6, 12$
$-1, -2, -3, -4, -6, -12$

$7 \nmid 12$ because there is no $k \in \mathbb{Z}$ with $12 = 7k$. This would need $k = \frac{12}{7} \notin \mathbb{Z}$

$-3 \mid 12$ because $12 = (-3)(-4)$
$b = ak$

$6 \mid 12$ because $12 = (6)(2)$
$b = ak$

Def: Let $p \in \mathbb{Z}$ with $p > 1$.
We say that p is a **prime** if the only positive divisors of p are 1 and p. If p is not a prime, then we call it a **composite** number.

Ex: Let's circle the primes...

not prime since $4 = (2)(2)$

not prime since $6 = (2)(3)$

2, 3, 4, 5, 6, 7, 8, 9, 10,
11, 12, 13, 14, 15, 16, 17, 18, 19,
20, 21, 22, 23, 24, 25, 26, 27, ...

Proposition: Let x and y be positive integers. If $x \mid y$, then $1 \leq x \leq y$.

proof: Suppose x and y are positive integers and that $x \mid y$.
Since x is a positive integer we know $1 \leq x$.
Since $x \mid y$ we know $y = xk$ where $k \in \mathbb{Z}$.

We know k is positive because $k = \frac{y}{x}$ and x and y are both positive.

using properties of the rational numbers $\mathbb{Q}$

Thus, $1 \leq k$.

multiply by x on both sides

Since x is positive this gives $x \leq \underbrace{xk}_{y}$.

So, $1 \leq x \leq xk = y$.

Thus, $1 \leq x \leq y$.

<u>Proposition:</u> Let p and q be prime numbers.
If $p \mid q$, then $p = q$.

<u>proof:</u> Suppose p and q are primes and $p \mid q$.

Since q is prime it's only positive divisors are 1 and q.

Since p is positive and $p \mid q$, then this means $p = 1$ or $p = q$.

Since p is prime, we know $p > 1$.

So, $p = q$.