

Math 4460

1/29/25



Let's look how the primes are spaced out...

(2), (3), 4, (5), 6, (7), 8, 9, 10, (11), 12, (13),
14, 15, 16, (17), 18, (19), 20, 21, 22,
(23), 24, 25, 26, 27, 28, (29), 30,
(31), 32, 33, 34, 35, 36, (37), 38,
39, 40, (41), 42, (43), 44, 45, 46, (47),
48, 49, 50, 51, 52, (53), 54, 55, 56,
57, 58, (59), 60, (61), 62, 63, 64, 65,
66, (67), 68, 69, 70, (71), 72, (73), 74,
75, 76, 77, 78, (79), 80, 81, 82,
(83), 84, 85, 86, 87, 88, (89), 90,
91, 92, 93, 94, 95, 96, (97), 98,
99, 100, (101), ...

Given $N > 0$, can we find N consecutive composite integers?

Example with $N=4$

$(N+1)! + 2, (N+1)! + 3, (N+1)! + 4, (N+1)! + 5$

$5! + 2, 5! + 3, 5! + 4, 5! + 5$

$\underbrace{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 + 2}_{2 \text{ divides this}}, \underbrace{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 + 3}_{3 \text{ divides this}}, \underbrace{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 + 4}_{4 \text{ divides this}}, \underbrace{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 + 5}_{5 \text{ divides this}}$

$\underbrace{122}_{2 \cdot 61}, \underbrace{123}_{3 \cdot 41}, \underbrace{124}_{4 \cdot 31}, \underbrace{125}_{5 \cdot 25}$

We just found 4 consecutive
composite integers.
not prime

Theorem: There are arbitrarily large gaps in the primes. That is, given $N > 0$ there exist N consecutive composite positive integers.

Proof: Given $N > 0$, consider the following consecutive positive integers:

$$(N+1)! + 2, (N+1)! + 3, (N+1)! + 4, \dots, (N+1)! + (N+1)$$

We need to show these are all composite.

Let $k \in \mathbb{Z}$ with $2 \leq k \leq N+1$.

Then,


$$\begin{aligned} (N+1)! + k &= \overbrace{(N+1)!}^{(N+1)!} + k \\ &= (N+1) \cdot N \cdot (N-1) \cdots (k+1) \cdot k \cdot (k-1) \cdots (2)(1) + k \\ &= k \left[(N+1) \cdot N \cdot (N-1) \cdots (k+1)(k-1) \cdots (2)(1) + 1 \right] \end{aligned}$$

$$\text{So, } k \mid [(N+1)! + k]$$

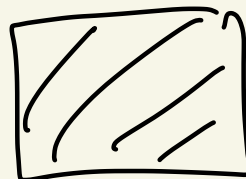
We know $k \neq 1$.

$$\text{And } k \leq N+1 < (N+1)! < (N+1)! + k$$

$$\text{So, } k \neq (N+1)! + k.$$

Thus, $(N+1)! + k$ is composite (not prime) 

So we have created N
consecutive composite positive
integers.



Ex: $N=8$

k	$(N+1)! + k$
2	362,882
3	362,883
4	362,884
5	362,885
6	362,886
7	362,887
8	362,888
9	362,889

HW 1

① Let $n > 1$ be an integer.

If $2^n - 1$ is prime,
then n is prime.

Ex: $2^{\textcircled{3}} - 1 = \textcircled{7}$ ← prime

Not sufficient — prime

② $2^{\textcircled{11}} - 1 = \textcircled{2047} = 23 \cdot 89$
not prime

(converse not true)

converse: If n is prime,
then $2^n - 1$ is prime

proof: We prove the contrapositive:

"If n is composite, then $2^n - 1$ is composite"

Suppose n is composite.

By HW 1 #7(b)

We get
 $n = ab$

where $a, b \in \mathbb{Z}$ and $1 < a, 1 < b$.

Then,

$$2^n - 1 = 2^{ab} - 1$$

$$= (2^a - 1) \left(2^{a(b-1)} + 2^{a(b-2)} + \dots + 2^a + 2 + 1 \right)$$

Ex:

$$a = 2$$

$$b = 4$$

$$\begin{aligned} 2^8 - 1 &= 2^{2 \cdot 4} - 1 \\ &= (2^2 - 1) \left(2^{2 \cdot 3} + 2^{2 \cdot 2} + 2^{2 \cdot 1} + 1 \right) \\ &= (2^2 - 1) \left(2^6 + 2^4 + 2^2 + 1 \right) \\ &= 2^8 + 2^6 + 2^4 + 2^2 \end{aligned}$$

If P , then Q
 \Updownarrow
If $\neg Q$, then $\neg P$

contrapositive

$$\begin{aligned} & -2^6 - 2^4 - 2^2 - 1 \\ & = 2^8 - 1 \end{aligned}$$

Since $a > 1$, we know

$$2^a - 1 \geq 2^2 - 1 = 3 > 1$$

Since $b > 1$, the second factor satisfies

$$2^{a(b-1)} + 2^{a(b-2)} + \dots + 2^{2a} + \underbrace{2^a + 1}_{\text{at least get these terms}}$$

$$> 2^a + 1 \geq 2^2 + 1 = 5 > 1$$

Thus, $2^n - 1 = kl$ where $k, l \in \mathbb{Z}$
 $k > 1, l > 1$.

So, $2^n - 1$ is composite.

