Math 4460
1/29/25

Let's look how the primes are spaced out...

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13,
14, 15, 16, 17, 18, 19, 20, 21, 22,
23, 24, 25, 26, 27, 28, 29, 30,
31, 32, 33, 34, 35, 36, 37, 38,
39, 40, 41, 42, 43, 44, 45, 46, 47,
48, 49, 50, 51, 52, 53, 54, 55, 56,
57, 58, 59, 60, 61, 62, 63, 64, 65,
66, 67, 68, 69, 70, 71, 72, 73, 74,
75, 76, 77, 78, 79, 80, 81, 82,
83, 84, 85, 86, 87, 88, 89, 90,
91, 92, 93, 94, 95, 96, 97, 98,
99, 100, 101, ...

Given $N>0$, can we find N consecutive composite integers?

$\boxed{\text{Example with } N=4}$

$(N+1)!+2,\ (N+1)!+3,\ (N+1)!+4,\ (N+1)!+5$

$5!+2,\ 5!+3,\ 5!+4,\ 5!+5$

$\underbrace{5\cdot4\cdot3\cdot2\cdot1+2}_{\text{2 divides this}},\ \underbrace{5\cdot4\cdot3\cdot2\cdot1+3}_{\text{3 divides this}},\ \underbrace{5\cdot4\cdot3\cdot2\cdot1+4}_{\text{4 divides this}},\ \underbrace{5\cdot4\cdot3\cdot2\cdot1+5}_{\text{5 divides this}}$

$\underbrace{122}_{2\cdot61},\ \underbrace{123}_{3\cdot41},\ \underbrace{124}_{4\cdot31},\ \underbrace{125}_{5\cdot25}$

We just found 4 consecutive $\underbrace{\text{composite}}_{\text{not prime}}$ integers.

Theorem: There are arbitrarily large gaps in the primes. That is, given $N>0$ there exist N consecutive composite positive integers.

Proof: Given $N>0$, consider the following consecutive positive integers:

$$(N+1)!+2,\ (N+1)!+3,\ (N+1)!+4,\ \dots,\ (N+1)!+(N+1)$$

We need to show these are all composite.

Let $k \in \mathbb{Z}$ with $2 \leq k \leq N+1$.

Then,

$$(N+1)!+k =$$

$$= \overbrace{(N+1)\cdot N\cdot(N-1)\cdots(k+1)\cdot k\cdot(k-1)\cdots(2)(1)}^{(N+1)!} + k$$

$$= k\left[(N+1)\cdot N\cdot(N-1)\cdots(k+1)(k-1)\cdots(2)(1)+1\right]$$

So, $k \mid [(N+1)! + k]$

We know $k \neq 1$.

And $k \leq N+1 < (N+1)! < (N+1)! + k$

So, $k \neq (N+1)! + k$.

Thus, $(N+1)! + k$ is composite (not prime)

So we have created N consecutive composite positive integers.

$\blacksquare$

Ex: $N = 8$

k	$(N+1)! + k$
2	362,882
3	362,883
4	362,884
5	362,885
6	362,886
7	362,887
8	362,888
9	362,889

Hw 1

11 Let $n > 1$ be an integer.
If $2^n - 1$ is prime,
then n is prime.

Ex: $2^{3} - 1 = 7$ ← prime
(3 ← prime)

Not sufficient
$2^{11} - 1 = \underbrace{2047}_{\text{not prime}} = 23 \cdot 89$
(11: prime)

(converse not true)

converse: If n is prime,
then $2^n - 1$ is prime

proof: We prove the contrapositive:
"If n is composite, then $2^n - 1$ is composite"

Suppose n is composite.

By HW 1 #7(b)

We get

$n = ab$

where $a, b \in \mathbb{Z}$ and $1 < a, 1 < b$.

If P, then Q

$\Updownarrow$

If $\neg Q$, then $\neg P$

contrapositive

Then,

$$2^n - 1 = 2^{ab} - 1$$

$$= (2^a - 1)\left(2^{a(b-1)} + 2^{a(b-2)} + \dots + 2^{2a} + 2^a + 1\right)$$

Ex:

$a = 2$
$b = 4$

$$2^8 - 1 = 2^{2 \cdot 4} - 1$$

$$= (2^2 - 1)\left(2^{2 \cdot 3} + 2^{2 \cdot 2} + 2^{2 \cdot 1} + 1\right)$$

$$= (2^2 - 1)\left(2^6 + 2^4 + 2^2 + 1\right)$$

$$= 2^8 + 2^6 + 2^4 + 2^2$$

$$-2^6 - 2^4 - 2^2 - 1$$
$$= 2^8 - 1$$

Since $a > 1$, we know
$$2^a - 1 \geq 2^2 - 1 = 3 > 1$$
Since $b > 1$, the second factor satisfies
$$2^{a(b-1)} + 2^{a(b-2)} + \dots + 2^{2a} + \underbrace{2^a + 1}_{\text{at least get these terms}}$$
$$> 2^a + 1 \geq 2^2 + 1 = 5 > 1$$

Thus, $2^n - 1 = k\ell$ where $k, \ell \in \mathbb{Z}$
$k > 1, \ell > 1$.

So, $2^n - 1$ is composite. $\blacksquare$