Math 4460
2/19/25

Proof of theorem from Monday

Let $a, b, c \in \mathbb{Z}$ where a and b are not both zero. Let $d = \gcd(a,b)$.

1

($\Rightarrow$) Suppose there exist integers x and y where

$$ax + by = c.$$

Our goal is to show that $d \mid c$.

Since $d = \gcd(a,b)$ we know that $d \mid a$ and $d \mid b$.

Thus, $a = dk$ and $b = dl$ where $k, l \in \mathbb{Z}$.

Plug these into $ax + by = c$

to get $dkx + dly = c$.

So, $d[\underbrace{kx + ly}_{\text{this is an integer}}] = c$

Therefore, $d \mid c$.

($\Leftarrow$)

Now assume that $d \mid c$.

Our goal is to find $x, y \in \mathbb{Z}$ where $ax + by = c$.

Since $d \mid c$ we know $c = dm$ where $m \in \mathbb{Z}$.

Since $d = \gcd(a, b)$ we know

$$ax_0 + by_0 = d$$

for some $x_0, y_0 \in \mathbb{Z}$.

use Euclidean algorithm

Multiply by m to get

$$a(mx_0) + b(my_0) = dm$$

This gives

$$ax + by = c$$

where $x = mx_0$, $y = my_0$.

$\boxed{1}$

2 Suppose $ax + by = c$ has integer solutions. Let's find a formula for all of them.

Let (x_0, y_0) be a particular solution, that is suppose

$$ax_0 + by_0 = c \quad \text{where } x_0, y_0 \in \mathbb{Z}$$

Our goal is to show that every solution is of the form

$$\left.\begin{aligned} x &= x_0 - t\left(\frac{b}{d}\right) \\ y &= y_0 + t\left(\frac{a}{d}\right) \end{aligned}\right\} \quad (*)$$

where $t \in \mathbb{Z}$

Let's first check that $(*)$ gives a solution by plugging it in. We get:

$$\begin{aligned} ax + by &= a\left(x_0 - t\left(\frac{b}{d}\right)\right) + b\left(y_0 + t\left(\frac{a}{d}\right)\right) \\ &= ax_0 - t\left(\frac{ab}{d}\right) + by_0 + t\left(\frac{ab}{d}\right) \\ &= ax_0 + by_0 \\ &= c \end{aligned}$$

But why does (*) give us <u>all</u> the solutions?

We are assuming $\boxed{ax_0 + by_0 = c.}$

Suppose we have another solution $\boxed{ax + by = c.}$

Subtract to get

$$a(x - x_0) + b(y - y_0) = 0$$

Thus,

$$b(y - y_0) = -a(x - x_0)$$

So,

$$b(y - y_0) = a(x_0 - x)$$

Thus,

$$\underbrace{\frac{b}{d}}(y - y_0) = \underbrace{\frac{a}{d}}(x_0 - x) \qquad (**)$$

$\frac{b}{d}, \frac{a}{d} \in \mathbb{Z}$ because $d|a$, $d|b$
$d = \gcd(a,b)$

(**) tells us that $\frac{a}{d} \mid \frac{b}{d}(y-y_0)$

From a previous theorem we know that $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

From another previous theorem since $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$ and $\frac{a}{d} \mid \frac{b}{d}(y-y_0)$ we know that $\frac{a}{d} \mid (y-y_0)$.

Thus, $y - y_0 = t\left(\frac{a}{d}\right)$ where $t \in \mathbb{Z}$.

So, $\boxed{y = y_0 + t\left(\frac{a}{d}\right)}$

Plug $y - y_0 = t\left(\frac{a}{d}\right)$ into $(**)$ to get:

$$\frac{b}{d}\left[t\left(\frac{a}{d}\right)\right] = \frac{a}{d}(x_0 - x)$$

So, $t\left(\frac{b}{d}\right) = x_0 - x$.

Thus, $\boxed{x = x_0 - t\left(\frac{b}{d}\right)}$

So every solution of $ax+by=c$ is of the form $(*)$.

2

3 Follows from 1 and 2.

QED

EX: (HW 2 #4(f))

Solve

$$\boxed{39x + 17y = 22}$$

Let

$d = \gcd(39, 17)$.

Euclidean algorithm time

$$\boxed{39} = 2 \cdot \boxed{17} + \boxed{5}$$
$$\boxed{17} = 3 \cdot \boxed{5} + \boxed{2}$$
$$\boxed{5} = 2 \cdot \boxed{2} + \boxed{1}$$
$$\boxed{2} = 2 \cdot \boxed{1} + 0$$

$\gcd(39,17) = 1$

Use above to solve $39x_0 + 17y_0 = 1$

Rewrite above with remainders on left side.

$$5 = 1\cdot\boxed{39} - 2\cdot\boxed{17}$$
$$2 = 1\cdot\boxed{17} - 3\cdot\boxed{5}$$
$$1 = 1\cdot\boxed{5} - 2\cdot\boxed{2}$$

start here

We get

$$1 = 1\cdot\boxed{5} - 2\cdot\boxed{2}$$
$$= 1\cdot\left(1\cdot\boxed{39} - 2\cdot\boxed{17}\right) - 2\cdot\left(1\cdot\boxed{17} - 3\cdot\boxed{5}\right)$$
$$= 1\cdot\boxed{39} - 4\cdot\boxed{17} + 6\cdot\boxed{5}$$
$$= 1\cdot\boxed{39} - 4\cdot\boxed{17} + 6\left(1\cdot\boxed{39} - 2\cdot\boxed{17}\right)$$
$$= 7\cdot\boxed{39} - 16\cdot\boxed{17}$$

Thus,

$$1 = 39(7) + 17(-16)$$

multiply by 22 to get:

$$22 = 39(154) + 17(-352)$$

Thus

$$39x_0 + 17y_0 = 22$$

where $x_0 = 154$, $y = -352$

Every solution to

$$39x + 17y = 22$$

is of the form

$$x = x_0 - t\left(\frac{b}{d}\right) = 154 - t\left(\frac{17}{1}\right)$$

$$y = y_0 + t\left(\frac{a}{d}\right) = -352 + t\left(\frac{39}{1}\right)$$

That is, any solution is of the form

$$x = 154 - 17t$$
$$y = -352 + 39t$$

where t is any integer