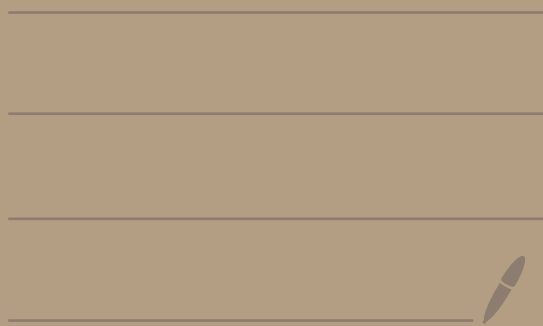


Math 4460

3/1/23



Test 1 is

Weds 3/8

---

We can cover any  
problems you want  
to do on Monday

---

Theorem Let  $n \in \mathbb{Z}$  with  $n \geq 2$   
Let  $w, x, y, z \in \mathbb{Z}$ . Then:

①  $x \equiv x \pmod{n}$

② If  $x \equiv y \pmod{n}$ ,  
then  $y \equiv x \pmod{n}$ .

③ If  $x \equiv y \pmod{n}$   
and  $y \equiv z \pmod{n}$ ,  
then  $x \equiv z \pmod{n}$ .

④ If  $w \equiv x \pmod{n}$  and  $y \equiv z \pmod{n}$ ,  
then  $w + y \equiv x + z \pmod{n}$   
and  $wy \equiv xz \pmod{n}$ .

Also,  $w - y \equiv x - z \pmod{n}$ .

⑤  $x \equiv y \pmod{n}$  iff

$x = y + nk$  where  $k \in \mathbb{Z}$ .

- ① reflexive
- ② symmetric
- ③ transitive

$\equiv$  is an  
equivalence  
relation

Proof:

$$\textcircled{1} \quad x - x = 0 = n(0)$$

$$\text{So, } n \mid (x - x).$$

$$\text{Thus, } x \equiv x \pmod{n}.$$

$a \equiv b \pmod{n}$   
means  
 $n \mid (a - b)$

$$\textcircled{2} \quad \text{Suppose } x \equiv y \pmod{n}.$$

$$\text{Then, } n \mid (x - y).$$

$$\text{So, } x - y = n\ell \text{ where } \ell \in \mathbb{Z}.$$

$$\text{Thus, } y - x = n(-\ell).$$

$$\text{Hence, } n \mid (y - x).$$

$$\text{So, } y \equiv x \pmod{n}.$$

$$\textcircled{3} \quad \text{Suppose } x \equiv y \pmod{n} \\ \text{and } y \equiv z \pmod{n}.$$

Then,  $n \mid (x-y)$  and  $n \mid (y-z)$ .

Hence,  $x-y = ns$  and  $y-z = nt$

where  $s, t \in \mathbb{Z}$ .

Adding gives

$$x - z = ns + nt$$

$$\text{So, } x - z = n[s + t].$$

Ergo  $n \mid (x-z)$ .

It follows that  $x \equiv z \pmod{n}$ .

Alternative  
method

$$\begin{aligned} x - z &= \\ (y + ns) & \\ - (y - nt) & \\ &= ns + nt \\ &= n(s + t) \end{aligned}$$

---

④ Suppose  $w \equiv x \pmod{n}$   
and  $y \equiv z \pmod{n}$ .

Then,  $n \mid (w-x)$  and  $n \mid (y-z)$ .

Thus,  $w-x = n\alpha$  and  $y-z = n\beta$

where  $\alpha, \beta \in \mathbb{Z}$ .

Then  $(w+y) - (x+z)$

$$= (w-x) + (y-z)$$

$$= n\alpha + n\beta$$

$$= n[\alpha + \beta]$$

So,  $n \mid [(w+y) - (x+z)]$

Thus,

$$(w+y) \equiv (x+z) \pmod{n}$$

Note that

$$\begin{aligned}w - x &= n\alpha \\ y - z &= n\beta\end{aligned}$$

$$wy - xz$$

$$= \underbrace{(x + n\alpha)}_w y - x \underbrace{(y - n\beta)}_z$$

$$= xy + n\alpha y - xy + xn\beta$$

$$= n[\alpha y + x\beta]$$

Hence,  $n \mid [wy - xz]$ .

So,  $wy \equiv xz \pmod{n}$ .

Also,

$$\begin{aligned}w-x &= n\alpha \\ y-z &= n\beta\end{aligned}$$

$$\begin{aligned}(w-y) - (x-z) & \\ &= (w-x) + (-y+z) \\ &= n\alpha - n\beta \\ &= n[\alpha - \beta].\end{aligned}$$

$$\text{So, } n \mid [(w-y) - (x-z)]$$

$$\text{Thus, } (w-y) \equiv (x-z) \pmod{n}.$$

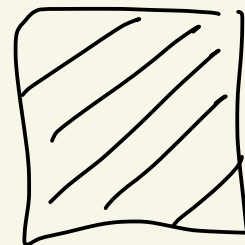


$$\textcircled{5} \quad x \equiv y \pmod{n}$$

$$\text{iff } n \mid (x - y)$$

$$\text{iff } x - y = nk \text{ for some } k \in \mathbb{Z}$$

$$\text{iff } x = y + nk \text{ for some } k \in \mathbb{Z}.$$



Def: Let  $n \in \mathbb{Z}$  with  $n \geq 2$ .

Let  $x \in \mathbb{Z}$ .

The equivalence class of  $x$   
modulo  $n$  is

$$\bar{x} = \{ y \in \mathbb{Z} \mid y \equiv x \pmod{n} \}$$

$$= \{ \dots, x-3n, x-2n, x-n, \\ x, x+n, x+2n, x+3n, \dots \}$$

⑤ from  
theorem

Ex:  $n=2$

$$\bar{0} = \{ y \in \mathbb{Z} \mid y \equiv 0 \pmod{2} \}$$

$$= \{ \dots, -8, -6, -4, -2, 0, 2, 4, 6, 8, \dots \}$$

$$\bar{1} = \{ y \in \mathbb{Z} \mid y \equiv 1 \pmod{2} \}$$

$$= \{ \dots, -7, -5, -3, -1, 1, 3, 5, 7, \dots \}$$

$$\bar{2} = \{ y \in \mathbb{Z} \mid y \equiv 2 \pmod{2} \}$$

$$= \{ \dots, -6, -4, -2, 0, 2, 4, 6, 8, \dots \}$$

$$= \bar{0}$$

$$\bar{3} = \{ \dots, -5, -3, -1, 1, 3, 5, 7, 9, \dots \}$$

$$= \bar{1}$$

Note  $\overline{3} = \overline{1}$  and  $3 \equiv 1 \pmod{2}$

$\overline{0} = \overline{2}$  and  $0 \equiv 2 \pmod{2}$

Mod 2 partitions  $\mathbb{Z}$  into  
two equivalence classes:  $\overline{0}, \overline{1}$

$\overline{0}$

$\overline{1}$

$\mathbb{Z}$

... , -10  
, -8, -6, -4  
, -2, 0, 2, 4,  
6, 8, 10, 12, ...

... , -11, -9,  
-7, -5, -3,  
-1, 1, 3, 5, 7,  
9, 11, 13, ...

evens

odds

Ex:  $n = 3$

$$\begin{aligned}\bar{0} &= \{y \in \mathbb{Z} \mid y \equiv 0 \pmod{3}\} \\ &= \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\}\end{aligned}$$

$$\begin{aligned}\bar{1} &= \{y \in \mathbb{Z} \mid y \equiv 1 \pmod{3}\} \\ &= \{\dots, -8, -5, -2, 1, 4, 7, 10, 13, \dots\}\end{aligned}$$

$$\begin{aligned}\bar{2} &= \{y \in \mathbb{Z} \mid y \equiv 2 \pmod{3}\} \\ &= \{\dots, -7, -4, -1, 2, 5, 8, 11, \dots\}\end{aligned}$$

$$\begin{aligned}\bar{3} &= \{y \in \mathbb{Z} \mid y \equiv 3 \pmod{3}\} \\ &= \{\dots, -6, -3, 0, 3, 6, 9, 12, \dots\}\end{aligned}$$

$= \bar{0}$  and note  $3 \equiv 0 \pmod{3}$

Mod 3 partitions  $\mathbb{Z}$  into  
3 equivalence classes:  $\bar{0}, \bar{1}, \bar{2}$

$\bar{0}$

...  
-9  
-6  
-3  
0  
3  
6  
9  
...

$\bar{1}$

...  
-8  
-5  
-2  
1  
4  
7  
10  
...

$\bar{2}$

...  
-7  
-4  
-1  
2  
5  
8  
...

$\mathbb{Z}$