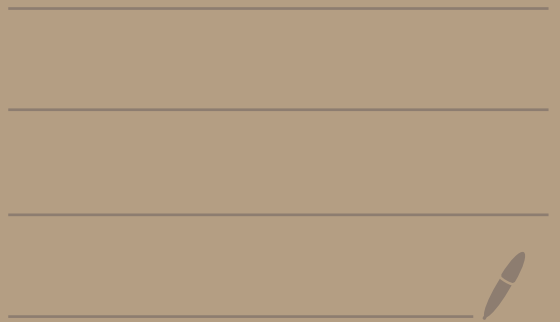


44/60
3/17/25



HW 2

(12) $a, b \in \mathbb{Z}$ not both zero
If there exist integers
 x, y with $ax + by = 1$,
then $\gcd(a, b) = 1$.

proof: Let $d = \gcd(a, b)$.

Then, $d|a$, $d|b$, and $d \geq 1$.

Thus, $a = dk$, $b = dl$
where $k, l \in \mathbb{Z}$.

We are given $ax + by = 1$.

Plugging in we get

$$dkx + dly = 1.$$

$$\text{So, } d[kx + ly] = 1.$$

$$\text{Then, } d \mid 1.$$

$$\text{So, } d = \pm 1.$$

$$\text{Since } d \geq 1 \text{ we get } d = 1.$$



HW 2

$$4(a) \quad 4001x + 2689y = 1$$

$$4001 = 1 \cdot 2689 + 1312$$

$$2689 = 2 \cdot 1312 + 65$$

$$1312 = 20 \cdot 65 + 12$$

$$65 = 5 \cdot 12 + 5$$

$$12 = 2 \cdot 5 + 2$$

$$5 = 2 \cdot 2 + \textcircled{1}$$

$$2 = 2 \cdot 1 + 0$$

gcd

$$d = \text{gcd}(4001, 2689) = 1$$

$$\begin{aligned}
 1312 &= 1 \cdot 4001 - 1 \cdot 2689 \\
 65 &= 1 \cdot 2689 - 2 \cdot 1312 \\
 12 &= 1 \cdot 1312 - 20 \cdot 65 \\
 5 &= 1 \cdot 65 - 5 \cdot 12 \\
 2 &= 1 \cdot 12 - 2 \cdot 5 \\
 1 &= 1 \cdot 5 - 2 \cdot 2
 \end{aligned}$$

$$1 = 1 \cdot 5 - 2 \cdot 2$$

$$= 1 \cdot (1 \cdot 65 - 5 \cdot 12) - 2 \cdot (1 \cdot 12 - 2 \cdot 5)$$

$$= 1 \cdot 65 - 7 \cdot 12 + 4 \cdot 5$$

$$= 1 \cdot (1 \cdot 2689 - 2 \cdot 1312)$$

$$- 7 \cdot (1 \cdot 1312 - 20 \cdot 65)$$

$$+ 4 \cdot (1 \cdot 65 - 5 \cdot 12)$$

$$= 1 \cdot 2689 - 9 \cdot 1312 + 144 \cdot 65 - 20 \cdot 12$$

$$= 1 \cdot 2689 - 9 \cdot (1 \cdot 4001 - 1 \cdot 2689) \\ + 144 \cdot (1 \cdot 2689 - 2 \cdot 1312) \\ - 20 \cdot (1 \cdot 1312 - 20 \cdot 65)$$

$$= 154 \cdot 2689 - 9 \cdot 4001 \\ - 308 \cdot 1312 + 400 \cdot 65$$

$$= 154 \cdot 2689 - 9 \cdot 4001 \\ - 308 \cdot (1 \cdot 4001 - 1 \cdot 2689) \\ + 400 \cdot (1 \cdot 2689 - 2 \cdot 1312)$$

$$= 862 \cdot 2689 - 317 \cdot 4001 \\ - 800(1312)$$

$$= 862 \cdot 2689 - 317 \cdot 4001 \\ - 800(1 \cdot 4001 - 1 \cdot 2689)$$

$$= 1662 \cdot 2689 - 1117 \cdot 4001$$

We have

$$2689 \underbrace{(1662)}_{x_0} + 4001 \underbrace{(-1117)}_{y_0} = 1$$

All solutions:

$$x = x_0 - t \left(\frac{b}{d} \right) = 1662 - t \left(\frac{4001}{1} \right)$$

$$y = y_0 + t \left(\frac{a}{d} \right) = -1117 + t \left(\frac{2689}{1} \right)$$

$$x = 1662 - 4001t$$

$$y = -1117 + 2689t$$

HW 2

(7) $a, b > 0$
 $d = \gcd(a, b)$

Prove: $a|b$ iff $d = a$

proof: Let $a > 0, b > 0$.

(\Rightarrow) Suppose $a|b$ and $d = \gcd(a, b)$.

We know $\underbrace{a|a}$ and $\underbrace{a|b}_{\text{given}}$.
 $a = (a)(1)$

So a is a positive common divisor of a and b .

Thus, $a \leq d$.

Also $d|a$ since $d = \gcd(a, b)$

and $a > 0, d > 0,$

thus we get $d \leq a$.

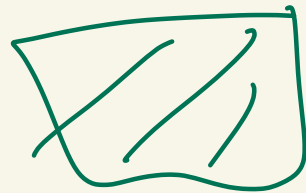
Since $a \leq d$ and $d \leq a$

we get $d = a$.

(\Leftarrow) Suppose $d = \gcd(a, b)$
and $d = a$.

Since $d = \gcd(a, b)$ we
know $d \mid b$.

Thus, $a \mid b$.



HW 2

⑧ Given: $\gcd(a, 4) = 2$, $\gcd(b, 4) = 2$

Show: $\gcd(a+b, 4) = 4$

proof:

Since $\gcd(a, 4) = 2$ we know $2 \mid a$.

So, $a = 2k$ where $k \in \mathbb{Z}$.

We can't have k be even because if it were then $k = 2c$ ($c \in \mathbb{Z}$)

and then $a = 2k = 4c$ and then $4 \mid a$ giving $\gcd(a, 4) = 4$ which it isn't.

Thus k is odd.

Similarly since $\gcd(b, 4) = 2$

We know $b = 2l$
where l is odd.

Then,

$$\begin{aligned} a+b &= 2k+2l \\ &= 2(k+l) = 2(2m) = 4m \end{aligned}$$

even
because
 k, l odd

$k+l=2m$
 $m \in \mathbb{Z}$

where $m \in \mathbb{Z}$.

Thus, $4 \mid (a+b)$.

So, $\gcd(a+b, 4) = 4$.

