$\boxed{\begin{array}{l} 4460 \\ 3/17/25 \end{array}}$

$\boxed{\text{HW 2}}$

12 $a, b \in \mathbb{Z}$ not both zero
If there exist integers x, y with $ax+by=1$, then $\gcd(a,b)=1$.

<u>Proof:</u> Let $d = \gcd(a,b)$.
Then, $d \mid a$, $d \mid b$, and $d \geq 1$.
Thus, $a = dk$, $b = d\ell$ where $k, \ell \in \mathbb{Z}$.
We are given $ax+by=1$.
Plugging in we get
$$dkx + d\ell y = 1.$$

So, $d[kx+ly]=1$.

Then, $d \mid 1$.

So, $d = \pm 1$.

Since $d \geq 1$ we get $d=1$. $\blacksquare$

$\boxed{\text{HW 2}}$

4(a) $4001x + 2689y = 1$

$$4001 = 1 \cdot 2689 + 1312$$
$$2689 = 2 \cdot 1312 + 65$$
$$1312 = 20 \cdot 65 + 12$$
$$65 = 5 \cdot 12 + 5$$
$$12 = 2 \cdot 5 + 2$$
$$5 = 2 \cdot 2 + 1 \quad \boxed{\text{gcd}}$$
$$2 = 2 \cdot 1 + 0$$

$$d = \gcd(4001, 2689) = 1$$

$$1312 = 1\cdot\boxed{4001} - 1\cdot\boxed{2689}$$
$$65 = 1\cdot\boxed{2689} - 2\cdot\boxed{1312}$$
$$12 = 1\cdot\boxed{1312} - 20\cdot\boxed{65}$$
$$5 = 1\cdot\boxed{65} - 5\cdot\boxed{12}$$
$$2 = 1\cdot\boxed{12} - 2\cdot\boxed{5}$$
$$1 = 1\cdot\boxed{5} - 2\cdot\boxed{2}$$

$$1 = 1\cdot\boxed{5} - 2\cdot\boxed{2}$$
$$= 1\cdot\left(1\cdot\boxed{65} - 5\cdot\boxed{12}\right) - 2\cdot\left(1\cdot\boxed{12} - 2\cdot\boxed{5}\right)$$
$$= 1\cdot\boxed{65} - 7\cdot\boxed{12} + 4\cdot\boxed{5}$$
$$= 1\cdot\left(1\cdot\boxed{2689} - 2\cdot\boxed{1312}\right)$$
$$- 7\cdot\left(1\cdot\boxed{1312} - 20\cdot\boxed{65}\right)$$

$$+4\cdot(1\cdot\boxed{65}-5\cdot\boxed{12})$$

$$=1\cdot\boxed{2689}-9\cdot\boxed{1312}+144\cdot\boxed{65}-20\cdot\boxed{12}$$

$$=1\cdot\boxed{2689}-9\cdot(1\cdot\boxed{4001}-1\cdot\boxed{2689})$$
$$+144\cdot(1\cdot\boxed{2689}-2\cdot\boxed{1312})$$
$$-20\cdot(1\cdot\boxed{1312}-20\cdot\boxed{65})$$

$$=154\cdot\boxed{2689}-9\cdot\boxed{4001}$$
$$-308\cdot\boxed{1312}+400\cdot\boxed{65}$$

$$=154\cdot\boxed{2689}-9\cdot\boxed{4001}$$
$$-308\cdot(1\cdot\boxed{4001}-1\cdot\boxed{2689})$$
$$+400\cdot(1\cdot\boxed{2689}-2\cdot\boxed{1312})$$

$$= 862 \cdot \boxed{2689} - 317 \cdot \boxed{4001}$$
$$- 800(1312)$$

$$= 862 \cdot \boxed{2689} - 317 \cdot \boxed{4001}$$
$$- 800(1 \cdot \boxed{4001} - 1 \cdot \boxed{2689})$$

$$= 1662 \cdot \boxed{2689} - 1117 \cdot \boxed{4001}$$

We have

$$\boxed{2689\underbrace{(1662)}_{x_0} + 4001\underbrace{(-1117)}_{y_0} = 1}$$

All solutions:

$$x = x_0 - t\left(\frac{b}{d}\right) = 1662 - t\left(\frac{4001}{1}\right)$$
$$y = y_0 + t\left(\frac{a}{d}\right) = -1117 + t\left(\frac{2689}{1}\right)$$

$$x = 1662 - 4001\,t$$
$$y = -1117 + 2689\,t$$

$\boxed{\text{HW 2}}$

(7) $a, b > 0$

$d = \gcd(a,b)$

Prove: $a \mid b$ iff $d = a$

---

proof: Let $a > 0$, $b > 0$.

$(\Rightarrow)$ Suppose $a \mid b$ and $d = \gcd(a,b)$.

We know $\underbrace{a \mid a}_{a = (a)(1)}$ and $\underbrace{a \mid b}_{\text{given}}$.

So a is a positive common divisor of a and b.

Thus, $\boxed{a \leq d.}$

Also $d \mid a$ since $d = \gcd(a,b)$

and $a>0$, $d>0$,

thus we get $\boxed{d \leq a}$.

Since $a \leq d$ and $d \leq a$

we get $d = a$.

($\Leftarrow$) Suppose $d = \gcd(a,b)$

and $\boxed{d = a}$.

Since $d = \gcd(a,b)$ we

know $\boxed{d \mid b}$.

Thus, $a \mid b$.

$\blacksquare$

HW 2

8 Given: $\gcd(a,4)=2$, $\gcd(b,4)=2$

Show: $\gcd(a+b,4)=4$

proof:

Since $\gcd(a,4)=2$ we know $2|a$.

So, $a=2k$ where $k\in\mathbb{Z}$.

We can't have k be even because if it were then $k=2c$ $(c\in\mathbb{Z})$ and then $a=2k=4c$ and then $4|a$ giving $\gcd(a,4)=4$ which it isn't.

Thus k is odd.

Similarly since $\gcd(b,4)=2$

We know $b = 2l$

where l is odd.

Then,

$$
\begin{aligned}
a+b &= 2k+2l \\
&= 2\underbrace{(k+l)}_{\text{even because } k,\, l \text{ odd}} = 2\underbrace{(2m)}_{k+l=2m,\ m\in\mathbb{Z}} = 4m
\end{aligned}
$$

where $m \in \mathbb{Z}$.

Thus, $4 \mid (a+b)$.

So, $\gcd(a+b, 4) = 4$.

$\blacksquare$