Math 4460

3/20/23

I'll have the tests
on Weds

Theorem: Let $n \in \mathbb{Z}$, $n \geq 2$.
Let $a, b, c \in \mathbb{Z}$.
In $\mathbb{Z}_n$ we have that:

$$\left.\begin{array}{l} ① \ \bar{a} + \bar{b} = \bar{b} + \bar{a} \\ ② \ \bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a} \end{array}\right\} \text{commutative properties}$$

$$\left.\begin{array}{l} ③ \ \bar{a} + (\bar{b} + \bar{c}) = (\bar{a} + \bar{b}) + \bar{c} \\ ④ \ \bar{a} \cdot (\bar{b} \cdot \bar{c}) = (\bar{a} \cdot \bar{b}) \cdot \bar{c} \end{array}\right\} \text{associative properties}$$

$$\left.\begin{array}{l} ⑤ \ \bar{a} \cdot (\bar{b} + \bar{c}) = \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c} \\ ⑥ \ (\bar{b} + \bar{c}) \cdot \bar{a} = \bar{b} \cdot \bar{a} + \bar{c} \cdot \bar{a} \end{array}\right\} \text{distributive properties}$$

proof: This is a HW 4 problem 11.
Let's do ⑥ to see how to do these.

We have that

$$(\bar{b}+\bar{c})\cdot\bar{a} = \overline{b+c}\cdot\bar{a}$$

def of + and · in $\mathbb{Z}_n$

$$= \overline{(b+c)a}$$

$a,b,c \in \mathbb{Z}$
So,
$(b+c)a$
$= ba+ca$

$$= \overline{ba+ca}$$

$$= \overline{ba} + \overline{ca}$$

$$= \bar{b}\cdot\bar{a} + \bar{c}\cdot\bar{a}$$

$\blacksquare$

Ex: $\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$

$\overline{0} = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$

$\overline{1} = \{\ldots, -7, -5, -3, -1, 1, 3, 5, 7, \ldots\}$

Given $x \in \mathbb{Z}$, then in $\mathbb{Z}_2$ we have:

$\overline{x} = \overline{0}$ iff x is even

$\overline{x} = \overline{1}$ iff x is odd

$\mathbb{Z}_2$ "detects"

even/odd-ness

parity

Ex: $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$

In $\mathbb{Z}_4$,

$\bar{0} = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$

$\bar{2} = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$

even integers

$\bar{1} = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$\bar{3} = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

odd integers

Given $x \in \mathbb{Z}$, in $\mathbb{Z}_4$ we have:

x is even iff $\bar{x} = \bar{0}$ or $\bar{x} = \bar{2}$

x is odd iff $\bar{x} = \bar{1}$ or $\bar{x} = \bar{3}$

x is odd iff $x \equiv 1 \pmod 4$ or $x \equiv 3 \pmod 4$

Topic 4.5 – Application to Pythagorean Triples

Consider the equation

$$\boxed{x^2 + y^2 = z^2}$$

We will find formulas for all the integer solutions.

Consider the integer solution $x=5, y=12, z=13$

You can get infinitely many solutions to $x^2+y^2=z^2$ by scaling any particular solution.

For example, set

$x=5k, y=12k, z=13k$

is a solution given any $k\in\mathbb{Z}$.

Why?

$$x^2+y^2=(5k)^2+(12k)^2$$
$$=5^2k^2+12^2k^2$$

$$= (5^2 + 12^2)k^2$$

$$= 13^2 k^2$$

$$= (13k)^2 = z^2$$

Some example solutions gotten by scaling $x=5, y=12, z=13$

k	$x=5k$	$y=12k$	$z=13k$
1	5	12	13
−2	−10	−24	−26
10	50	120	130
⋮	⋮	⋮	⋮

Another way to get more solutions is to change the signs of a solution.

Ex: Sols to $x^2+y^2=z^2$

$(5,12,13)$, $(5,12,-13)$, $(5,-12,-13)$
$(5,-12,13)$, $(-5,12,13)$, $(-5,12,-13)$
$(-5,-12,13)$, $(-5,-12,-13)$

Another way to get solutions is to set one or all to be 0.

For example,

$x=2, y=0, z=2$

$x=0, y=5, z=-5$

$x = 0,\ y = 0,\ z = 0$

Def: We call (x, y, z) a <u>Pythagorean triple</u> if

① $x, y, z \in \mathbb{Z}$

② $(x, y, z) \neq (0, 0, 0)$ } not all zero

and ③ $x^2 + y^2 = z^2$

If (x, y, z) is a Pythagorean triple, we say that it is <u>positive</u> if $x > 0,\ y > 0,\ z > 0$

Ex: $(3,4,5)$ is a positive Pythagorean triple

Ex: $(0,2,-2)$, $(-3,-4,5)$ are Pythagorean triples

Ex: $(25,60,-65)$ is a Pythagorean triple because

$$25^2+60^2=625+3600=4225$$

$$(-65)^2=4225$$

and so

$$25^2+60^2=(-65)^2$$

Let $d = \gcd(25, 60, -65) = 5$

Then,

$$(25, 60, -65)$$
$$= (5 \cdot 5, 5 \cdot 12, -5 \cdot 13)$$
$$= (d \cdot 5, d \cdot 12, -d \cdot 13)$$

and $(5, 12, 13)$ is a positive Pythagorean triple with $\gcd(5, 12, 13) = 1$.

<u>Def:</u> Let (x, y, z) be a Pythagorean triple. We say that (x, y, z) is <u>primitive</u> if $\gcd(x, y, z) = 1$.

Theorem: Any Pythagorean triple is of the form $(\pm da, \pm db, \pm dc)$ where (a,b,c) is a primitive, Pythagorean triple and $a \geq 0, b \geq 0, c \geq 0$ and d is a positive integer.

$\boxed{\text{Ex:}}$ $(9,-12,-15) \leftarrow$ Pythagorean triple

$$(9,-12,-15) = (3\cdot 3, -3\cdot 4, -3\cdot 5)$$

$$d=3, \ (a,b,c) = (3,4,5)$$

proof of theorem: Let (x,y,z) be a Pythagorean triple.

Then $x^2+y^2=z^2$ and $(x,y,z) \neq (0,0,0)$

Let $d = \gcd(x, y, z)$.

From class, $\gcd\left(\frac{x}{d}, \frac{y}{d}, \frac{z}{d}\right) = 1$.

Set

$$a = \left|\frac{x}{d}\right|, \quad b = \left|\frac{y}{d}\right|, \quad c = \left|\frac{z}{d}\right|$$

Then,

$$(x, y, z) = (\pm da, \pm db, \pm dc)$$

The +/- sign depends on the signs of x, y, z.

We see $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$

Since $d \mid x$, $d \mid y$, $d \mid z$ we have a, b, c are integers.

Also, $\gcd(a, b, c) = 1$

And

$$a^2+b^2 = \left|\frac{x}{d}\right|^2 + \left|\frac{y}{d}\right|^2$$

$$= \frac{x^2}{d^2} + \frac{y^2}{d^2}$$

$$= \frac{1}{d^2}\left(x^2+y^2\right)$$

$$= \frac{1}{d^2} z^2$$

$$= \frac{z^2}{d^2}$$

$$= \left|\frac{z}{d}\right|^2$$

$$= c^2$$

So, (a,b,c) is a primitive Pythagorean triple. $\blacksquare$