

Math 4460

4/23/25



# HW 3 (modified)

Prove that  $\sqrt{\frac{2}{3}}$  is irrational.

proof: Suppose  $\sqrt{\frac{2}{3}}$  is rational.

Then,  $\sqrt{\frac{2}{3}} = \frac{a}{b}$  where  $a, b \in \mathbb{Z}$ ,

$b \neq 0$ , and  $\gcd(a, b) = 1$ .

$$\text{So, } \frac{2}{3} = \frac{a^2}{b^2}.$$

$$\text{Then, } 2b^2 = 3a^2.$$

$$\text{Then, } 2 \mid 3a^2.$$

2 is prime and  $2 \mid 3 \cdot a^2$

Pick 2 or 3  
to use

p prime  
p | xy  
then  
p | x or  
p | y

So either  $2|3$  or  $2|a^2$ .

Since  $2 \nmid 3$  we get  $2|a^2$ .

Then, 2 is prime and  $2|a \cdot a$ .

So,  $2|a$ .

Thus,  $a = 2k$  where  $k \in \mathbb{Z}$ .

Plug this back into  $2b^2 = 3a^2$   
to get

$$2b^2 = 3(2k)^2$$

$$\text{So, } 2b^2 = 3 \cdot 2 \cdot 2 \cdot k^2$$

$$\text{Then } b^2 = 3 \cdot 2 \cdot k^2.$$

$$\text{So, } 2|b^2$$

Since 2 is prime and  $2 \mid b \cdot b$   
we have that  $2 \mid b$ .

Hence  $2 \mid a$  and  $2 \mid b$ .

But then  $\gcd(a, b) \geq 2$

contradicting  $\gcd(a, b) = 1$ .

Thus,  $\sqrt{\frac{2}{3}}$  is irrational.

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HW 3

⑤

Let  $x, y$  be positive integers.

If  $4 \mid xy$  but  $4 \nmid x$ , then  $2 \mid y$ .

Proof: Suppose  $4 \mid xy$  and  $4 \nmid x$ .

We want to show that  $2 \mid y$ .

Let's assume  $2 \nmid y$  and show this leads to a contradiction.

Have  $xy = 4k$  where  $k \in \mathbb{Z}$ .

Thus,  $xy = 2(2k)$ .

So,  $2 \mid xy$ .

Since 2 is prime either  $2 \mid x$  or  $2 \mid y$ .

Since  $2 \nmid y$  we have  $2 \mid x$ .

Then  $x = 2l$  where  $l \in \mathbb{Z}$ .

Since  $4 \nmid x$  we must have  
that  $l$  is odd. ] So  $2 \nmid l$

Then  $\underbrace{(2l)y}_{xy} = 4k$

So,  $ly = 2k$

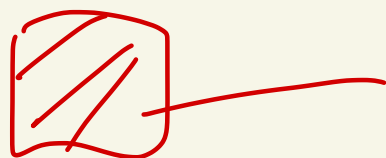
Then,  $2 \mid ly$ .

Since 2 is prime either  $2 \mid l$  or  $2 \mid y$ .

But  $2 \nmid l$  and  $2 \nmid y$ .

Contradiction.

Thus,  $2 \mid y$ .



## Method 2:

Suppose  $4 \mid xy$  and  $4 \nmid x$ .

Consider  $d = \gcd(4, y)$ .  $\leftarrow \begin{matrix} d=1, 2, \\ \text{or } 4 \end{matrix}$

If  $d=1$ ,

then  $4 \mid xy$  and  $\gcd(4, y)=1$ .

So, by a theorem from class,  $4 \mid x$ .

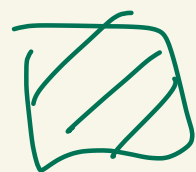
But  $4 \nmid x$ .

Thus,  $d \neq 1$ .

So,  $d=2$  or  $d=4$ .

So either  $2 \mid y$  or  $4 \mid y$ .

In either case  $2 \mid y$ .



HW 4  
13

Prove that  $15x^2 - 7y^2 = 1$   
has no integer solutions.

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proof: Suppose there exist  
 $x, y \in \mathbb{Z}$  where  $15x^2 - 7y^2 = 1$ .

Then in  $\mathbb{Z}_3$  we get

$$\overline{15} \bar{x}^2 + \overline{-7} \bar{y}^2 = \overline{1}$$

$\bar{0} \text{ in } \mathbb{Z}_3$        $\bar{2} \text{ in } \mathbb{Z}_3$

So,

$$\bar{2} \bar{y}^2 = \overline{1} \text{ in } \mathbb{Z}_3.$$

But this impossible  
by this table  $\rightarrow$

$\bar{y}$	$\bar{2} \bar{y}^2$
$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{2}(\bar{1})^2 = \bar{2}$
$\bar{2}$	$\bar{8} = \bar{2}$

} no  $\bar{1}$  here



Contradiction.

There are no  $x, y \in \mathbb{Z}$  where  
 $15x^2 - 7y^2 = 1$



## HW 4

(9) (a) Let  $p, x, y \in \mathbb{Z}$  where  $p$  is prime. Suppose  $\bar{x}\bar{y} = \bar{0}$  in  $\mathbb{Z}_p$ . Prove that  $\bar{x} = \bar{0}$  or  $\bar{y} = \bar{0}$  in  $\mathbb{Z}_p$ .

Proof:

Suppose  $\bar{x}\bar{y} = \bar{0}$  in  $\mathbb{Z}_p$ .

Then,  $xy \equiv 0 \pmod{p}$

Then,  $p \mid (xy - 0)$

Thus,  $p \mid xy$ .

Since  $p$  is prime,  $p \mid x$  or  $p \mid y$ .

Then,  $\underbrace{x \equiv 0 \pmod{p}}_{p \mid (x-0)}$  or  $\underbrace{y \equiv 0 \pmod{p}}_{p \mid (y-0)}$ .

So,  $\bar{x} = \bar{0}$  or  $\bar{y} = \bar{0}$  in  $\mathbb{Z}_p$ .



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9(b) Give an example in  $\mathbb{Z}_n$   
where  $\bar{x}\bar{y} = \bar{0}$  but  $\bar{x} \neq \bar{0}$   
and  $\bar{y} \neq \bar{0}$ .

In  $\mathbb{Z}_{14}$ ,  $\overbrace{\bar{2}}^{\bar{x}} \cdot \overbrace{\bar{7}}^{\bar{y}} = \bar{14} = \bar{0}$   
but  $\bar{2} \neq \bar{0}$  and  $\bar{7} \neq \bar{0}$ .