Math 4460

4/23/25

HW 3 2(b) (modified)

Prove that $\sqrt{\frac{2}{3}}$ is irrational.

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__proof:__ Suppose $\sqrt{\frac{2}{3}}$ is rational.

Then, $\sqrt{\frac{2}{3}} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$,

$b \neq 0$, and $\gcd(a,b) = 1$.

So, $\frac{2}{3} = \frac{a^2}{b^2}$.

Then, $\boxed{2b^2 = 3a^2}$.

Pick 2 or 3 to use

Then, $2 \mid 3a^2$.

2 is prime and $2 \mid 3 \cdot a^2$

p prime
$p \mid xy$
then
$p \mid x$ or
$p \mid y$

So either $2 \mid 3$ or $2 \mid a^2$.

Since $2 \nmid 3$ we get $2 \mid a^2$.

Then, 2 is prime and $2 \mid a \cdot a$.

$\boxed{\text{So, } 2 \mid a.}$

Thus, $\boxed{a = 2k}$ where $k \in \mathbb{Z}$.

Plug this back into $2b^2 = 3a^2$ to get

$$2b^2 = 3(2k)^2$$

So, $2b^2 = 3 \cdot 2 \cdot 2 \cdot k^2$

Then $b^2 = 3 \cdot 2 \cdot k^2$.

So, $2 \mid b^2$

Since 2 is prime and $2 \mid b \cdot b$
we have that $2 \mid b$.

Hence $2 \mid a$ and $2 \mid b$.
But then $\gcd(a,b) \geq 2$
contradicting $\gcd(a,b) = 1$.
Thus, $\sqrt{\frac{2}{3}}$ is irrational. $\blacksquare$

HW 3 5

Let $x, y$ be positive integers.

If $4 \mid xy$ but $4 \nmid x$, then $2 \mid y$

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proof: $\boxed{\text{Suppose } 4 \mid xy \text{ and } 4 \nmid x.}$

We want to show that $2 \mid y$.

Let's assume $\boxed{2 \nmid y}$ and show this leads to a contradiction.

Have $xy = 4k$ where $k \in \mathbb{Z}$.

Thus, $xy = 2(2k)$.

So, $2 \mid xy$.

Since 2 is prime either $2 \mid x$ or $2 \mid y$.

Since $2 \nmid y$ we have $2 \mid x$.

Then $x = 2\ell$ where $\ell \in \mathbb{Z}$.

Since $4 \nmid x$ we must have that $\ell$ is odd. So $2 \nmid \ell$

Then $\underbrace{(2\ell)y}_{xy} = 4k$

So, $\ell y = 2k$

Then, $2 \mid \ell y$.

Since 2 is prime either $2 \mid \ell$ or $2 \mid y$.

But $2 \nmid \ell$ and $2 \nmid y$.

Contradiction.

Thus, $2 \mid y$.

$\blacksquare$

## Method 2:

Suppose $4 \mid xy$ and $4 \nmid x$.

Consider $d = \gcd(4, y)$. ← $d = 1, 2,$ or $4$

If $d = 1$,

then $4 \mid xy$ and $\gcd(4, y) = 1$.

So, by a theorem from class, $4 \mid x$.

But $4 \nmid x$.

Thus, $d \neq 1$.

So, $d = 2$ or $d = 4$.

So either $2 \mid y$ or $4 \mid y$.

In either case $2 \mid y$. $\blacksquare$

$\boxed{\text{HW 4} \quad 13}$ Prove that $15x^2 - 7y^2 = 1$ has no integer solutions.

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<u>proof:</u> Suppose there exist $x, y \in \mathbb{Z}$ where $15x^2 - 7y^2 = 1$.

Then in $\mathbb{Z}_3$ we get

$$\underbrace{\overline{15}}_{\bar{0} \text{ in } \mathbb{Z}_3} \bar{x}^2 + \underbrace{\overline{-7}}_{\bar{2} \text{ in } \mathbb{Z}_3} \bar{y}^2 = \bar{1}$$

So,

$$\bar{2}\bar{y}^2 = \bar{1} \text{ in } \mathbb{Z}_3.$$

But this impossible by this table

| $\bar{y}$ | $\bar{2}\bar{y}^2$ |
|---|---|
| $\bar{0}$ | $\bar{0}$ |
| $\bar{1}$ | $\bar{2}(\bar{1})^2 = \bar{2}$ |
| $\bar{2}$ | $\bar{8} = \bar{2}$ |

no $\bar{1}$ here

Contradiction.

There are no $x, y \in \mathbb{Z}$ where

$$15x^2 - 7y^2 = 1$$

$\boxed{\text{HW 4}}$

9 (a) Let $p, x, y \in \mathbb{Z}$ where $p$ is prime. Suppose $\bar{x}\,\bar{y} = \bar{0}$ in $\mathbb{Z}_p$. Prove that $\bar{x} = \bar{0}$ or $\bar{y} = \bar{0}$ in $\mathbb{Z}_p$.

Proof:

Suppose $\bar{x}\,\bar{y} = \bar{0}$ in $\mathbb{Z}_p$.

Then, $xy \equiv 0 \pmod{p}$

Then, $p \mid (xy - 0)$

Thus, $p \mid xy$.

Since $p$ is prime, $p \mid x$ or $p \mid y$.

Then, $\underbrace{x \equiv 0 \pmod{p}}_{p \mid (x-0)}$ or $\underbrace{y \equiv 0 \pmod{p}}_{p \mid (y-0)}$.

So, $\overline{x} = \overline{0}$ or $\overline{y} = \overline{0}$ in $\mathbb{Z}_p$. $\blacksquare$

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$\boxed{9(b)}$ Give an example in $\mathbb{Z}_n$ where $\overline{x}\,\overline{y} = \overline{0}$ but $\overline{x} \neq \overline{0}$ and $\overline{y} \neq \overline{0}$.

In $\mathbb{Z}_{14}$, $\overbrace{\overline{2}}^{\overline{x}} \cdot \overbrace{\overline{7}}^{\overline{y}} = \overline{14} = \overline{0}$

but $\overline{2} \neq \overline{0}$ and $\overline{7} \neq \overline{0}$.