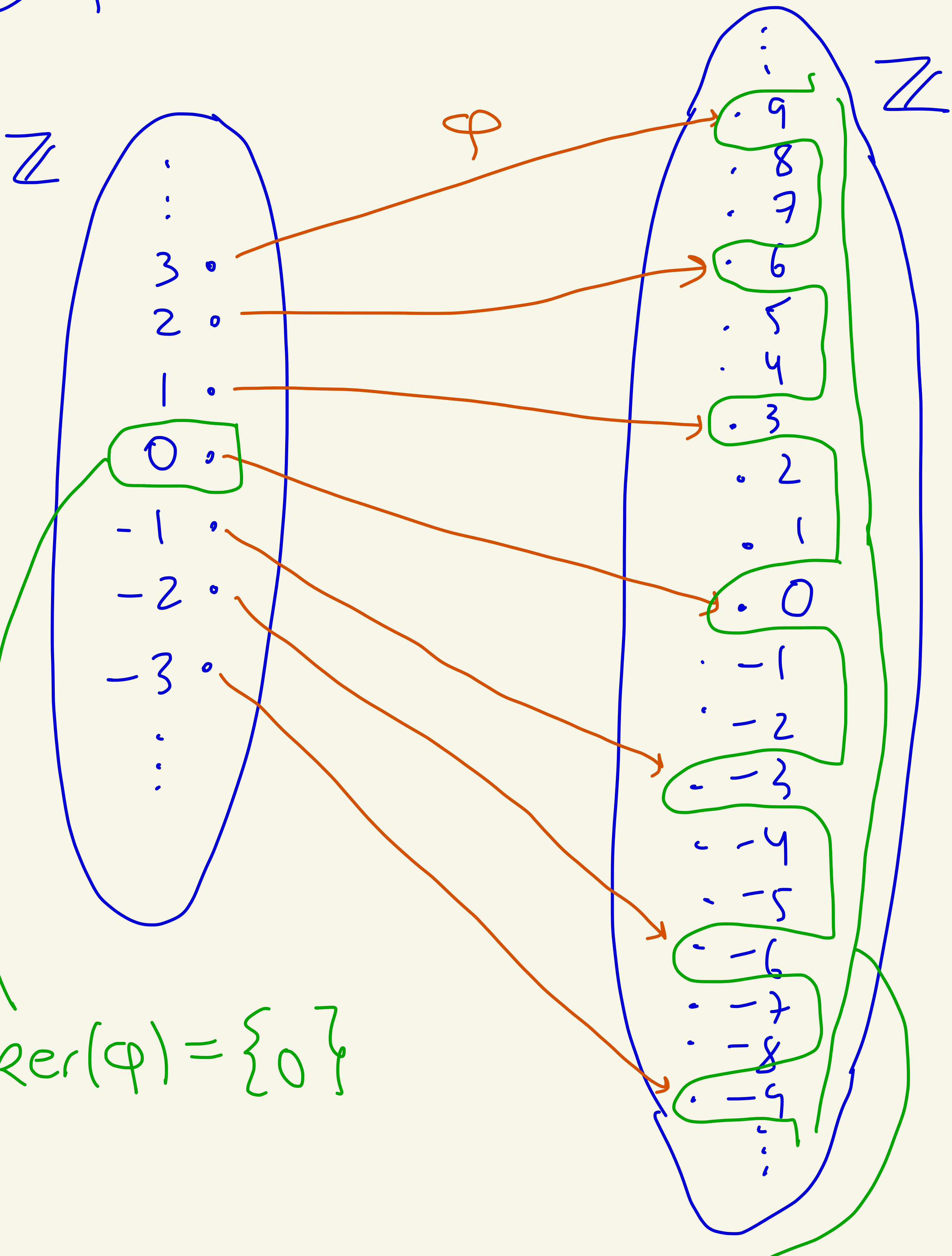
Math 4550

11/10/25

$\boxed{\text{HW 4}}$

① $\varphi : \mathbb{Z} \to \mathbb{Z}, \quad \varphi(n) = 3n$



$\ker(\varphi) = \{0\}$

$\text{im}(\varphi) = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$ ✓

φ is a homomorphism:

Let $a, b \in \mathbb{Z}$.

Then,

$$\varphi(a+b) = 3(a+b)$$
$$= 3a + 3b$$
$$= \varphi(a) + \varphi(b)$$

$\blacksquare$

φ is one-to-one:

Suppose $\varphi(a) = \varphi(b)$ where $a, b \in \mathbb{Z}$.

Then, $3a = 3b$.

So, $a = b$.

φ is not onto $\mathbb{Z}$:

$1 \in \mathbb{Z}$ but $1 \notin im(\varphi)$.

There is no $a \in \mathbb{Z}$ with

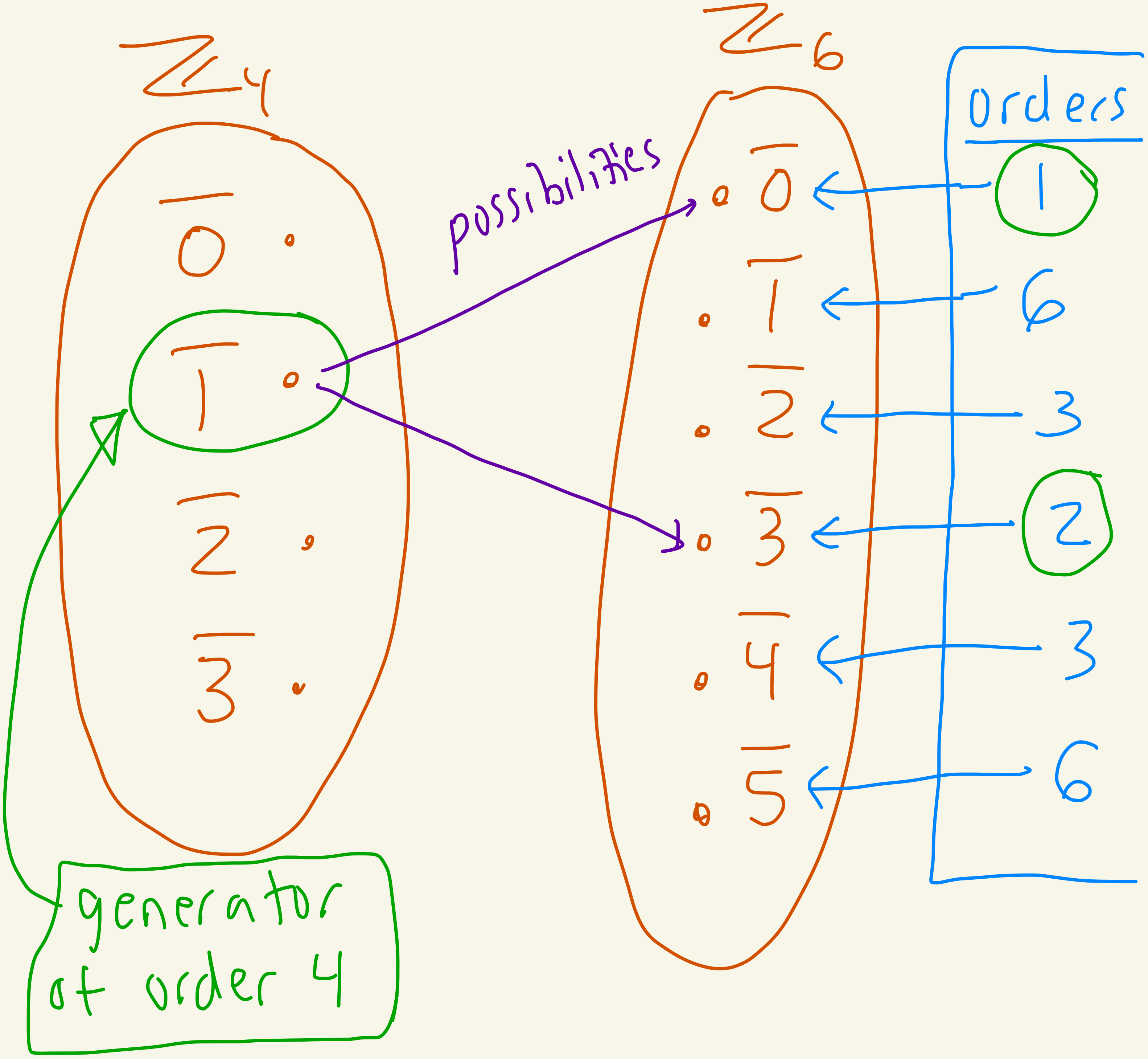
$$3a = \varphi(a) = 1.$$

Need $a = \frac{1}{3} \notin \mathbb{Z}$

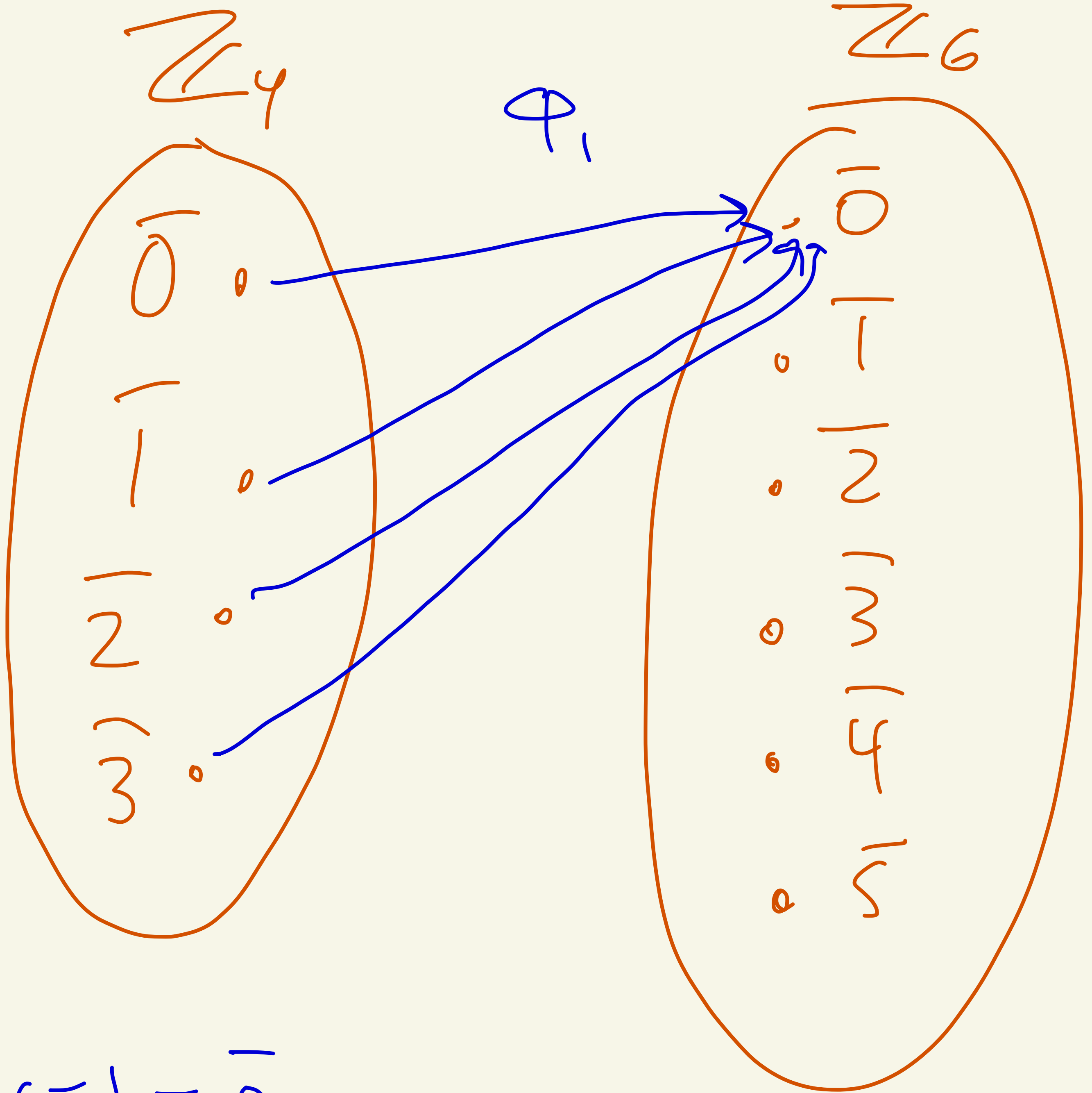
$\boxed{\text{HW 5-4}}$

Find all homomorphisms

$$\varphi : \mathbb{Z}_4 \to \mathbb{Z}_6$$



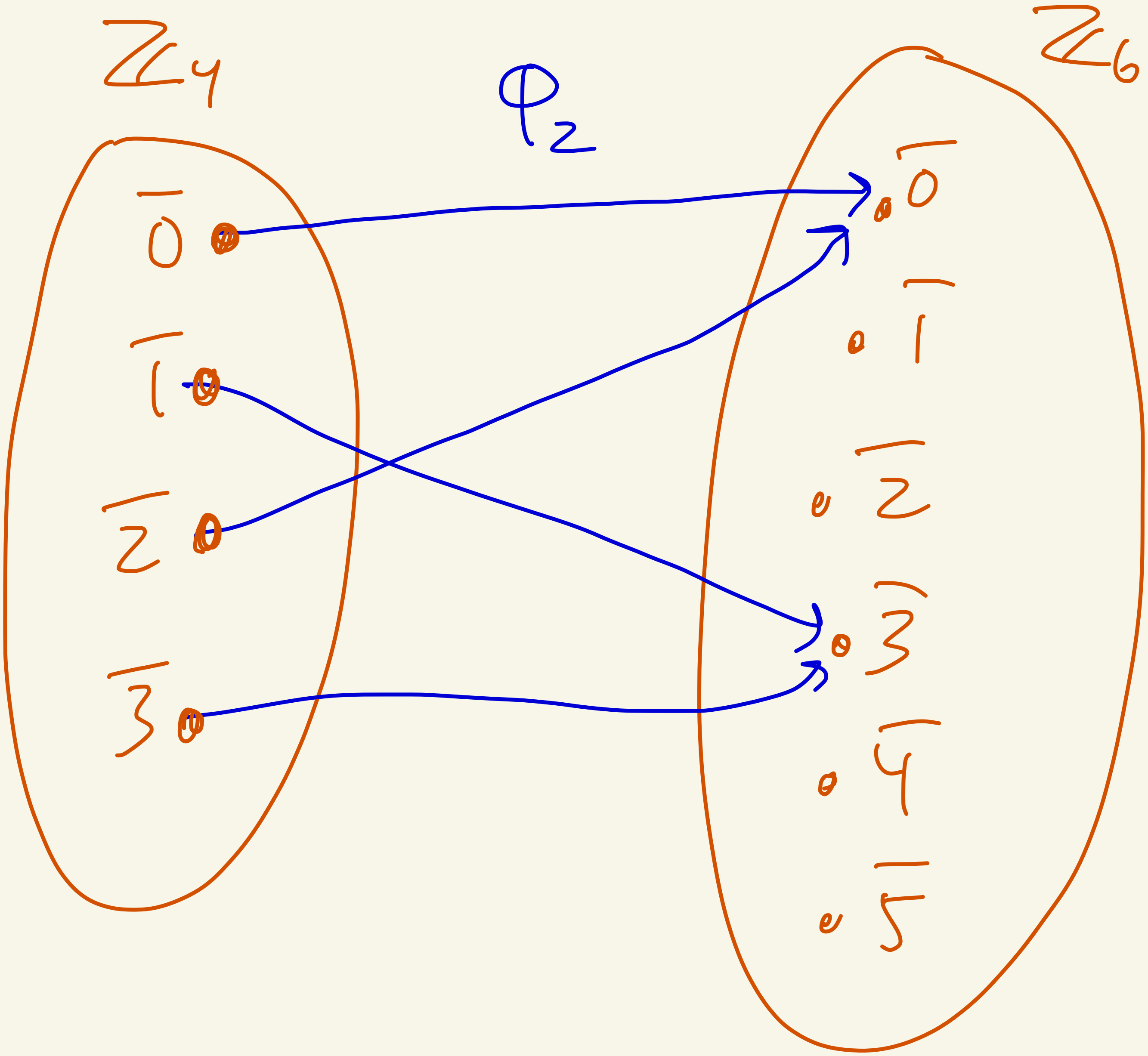
$\mathbb{Z}_4$
$\overline{0}$
$\overline{1}$
$\overline{2}$
$\overline{3}$
possibilities
$\mathbb{Z}_6$
$\overline{0}$
$\overline{1}$
$\overline{2}$
$\overline{3}$
$\overline{4}$
$\overline{5}$
orders
1
6
3
2
3
6
generator of order 4



$$\varphi_1(\bar{1}) = \bar{0}$$

$$\varphi_1(\bar{2}) = \varphi_1(\bar{1}+\bar{1}) = \varphi_1(\bar{1}) + \varphi_1(\bar{1})$$
$$= \bar{0} + \bar{0} = \bar{0}$$

$$\varphi_1(\bar{3}) = \varphi_1(\bar{1}) + \varphi_1(\bar{1}) + \varphi_1(\bar{1}) = \bar{0}+\bar{0}+\bar{0} = \bar{0}$$



$$\varphi_2(\bar{1}) = \bar{3}$$

$$\varphi_2(\bar{2}) = \varphi_2(\bar{1}+\bar{1}) = \varphi_2(\bar{1}) + \varphi_2(\bar{1})$$
$$= \bar{3} + \bar{3} = \bar{6} = \bar{0}$$

$$\varphi_2(\bar{3}) = \varphi_2(\bar{1}) + \varphi_2(\bar{1}) + \varphi_2(\bar{1})$$
$$= \bar{3} + \bar{3} + \bar{3} = \bar{9} = \bar{3}$$

$\boxed{\text{HW 6 - \#2}}$

$G = \mathbb{Z}_{12} = \{\cancel{\bar{0}}, \cancel{\bar{1}}, \cancel{\bar{2}}, \cancel{\bar{3}}, \cancel{\bar{4}}, \cancel{\bar{5}}, \cancel{\bar{6}}, \cancel{\bar{7}}, \cancel{\bar{8}}, \cancel{\bar{9}}, \cancel{\overline{10}}, \cancel{\overline{11}}\}$

$H = \langle \bar{4} \rangle = \{\bar{0}, \bar{4}, \bar{8}\}$

(a) Find the elements of G/H.

$\bar{0} + H = \{\bar{0}, \bar{4}, \bar{8}\} = \bar{4} + H = \bar{8} + H$

$\bar{1} + H = \{\bar{1}, \bar{5}, \bar{9}\} = \bar{5} + H = \bar{9} + H$

$\bar{2} + H = \{\bar{2}, \bar{6}, \overline{10}\} = \bar{6} + H = \overline{10} + H$

$\bar{3} + H = \{\bar{3}, \bar{7}, \overline{11}\} = \bar{7} + H = \overline{11} + H$

There are 4 left-cosets.

$G/H = \{\underbrace{\bar{0} + H}_{\text{identity element}}, \bar{1} + H, \bar{2} + H, \bar{3} + H\}$

$$(\bar{2}+H)+(\bar{3}+H) = (\bar{2}+\bar{3})+H$$
$$= \bar{5}+H$$
$$= \bar{1}+H$$

What's the inverse of $\bar{2}+H$?

$(\bar{2}+H)+(\bar{1}+H) = \bar{3}+H \leftarrow$ not identity

$(\bar{2}+H)+(\bar{2}+H) = \bar{4}+H = \bar{0}+H$

So, $(\bar{2}+H)^{-1} = \bar{2}+H$

$(\bar{1}+H)+(\bar{3}+H) = \bar{4}+H = \bar{0}+H$

So, $(\bar{1}+H)^{-1} = \bar{3}+H$.

G/H	$\bar{0}+H$	$\bar{1}+H$	$\bar{2}+H$	$\bar{3}+H$
$\bar{0}+H$	$\bar{0}+H$	$\bar{1}+H$	$\bar{2}+H$	$\bar{3}+H$
$\bar{1}+H$	$\bar{1}+H$	$\bar{2}+H$	$\bar{3}+H$	$\bar{0}+H$
$\bar{2}+H$	$\bar{2}+H$	$\bar{3}+H$	$\bar{0}+H$	$\bar{1}+H$
$\bar{3}+H$	$\bar{3}+H$	$\bar{0}+H$	$\bar{1}+H$	$\bar{2}+H$

$$G/H = \{\bar{0}+H, \bar{1}+H, \bar{2}+H, \bar{3}+H\}$$

$\bar{0}+H$	$\bar{1}+H$	$\bar{2}+H$	$\bar{3}+H$
$\bar{4}+H$	$\bar{5}+H$	$\bar{6}+H$	$\bar{7}+H$
$\bar{8}+H$	$\bar{9}+H$	$\overline{10}+H$	$\overline{11}+H$

$\bar{1}+H$ generates G/H.

$\boxed{\bar{1}+H}$

$(\bar{1}+H)+(\bar{1}+H)=\boxed{\bar{2}+H}$

$(\bar{1}+H)+(\bar{1}+H)+(\bar{1}+H)=\boxed{\bar{3}+H}$

$(\bar{1}+H)+(\bar{1}+H)+(\bar{1}+H)+(\bar{1}+H)=\bar{4}+H=\boxed{\bar{0}+H}$

So, G/H is cyclic. $\boxed{\bar{1}+H \text{ has order } 4}$

order of $\bar{2}+H$:

$\bar{2}+H\neq\bar{0}+H$

$(\bar{2}+H)+(\bar{2}+H)=\bar{4}+H=\bar{0}+H$

$\bar{2}+H$ has order 2

HW 6 #7 | G is a group, $|G|=n$
If $x \in G$, then $x^n = e$.

proof:
From class, the order of x must divide $|G|=n$.
Let the order of x be m.
Then, $n = mk$ where $k \in \mathbb{Z}$.

Then,
$$x^n = x^{mk} = (x^m)^k$$
$$= e^k = e$$

$\blacksquare$

$\boxed{\text{HW 4}}$

$\boxed{5(c)}$

Let $\varphi : G_1 \to G_2$ be a homomorphism.

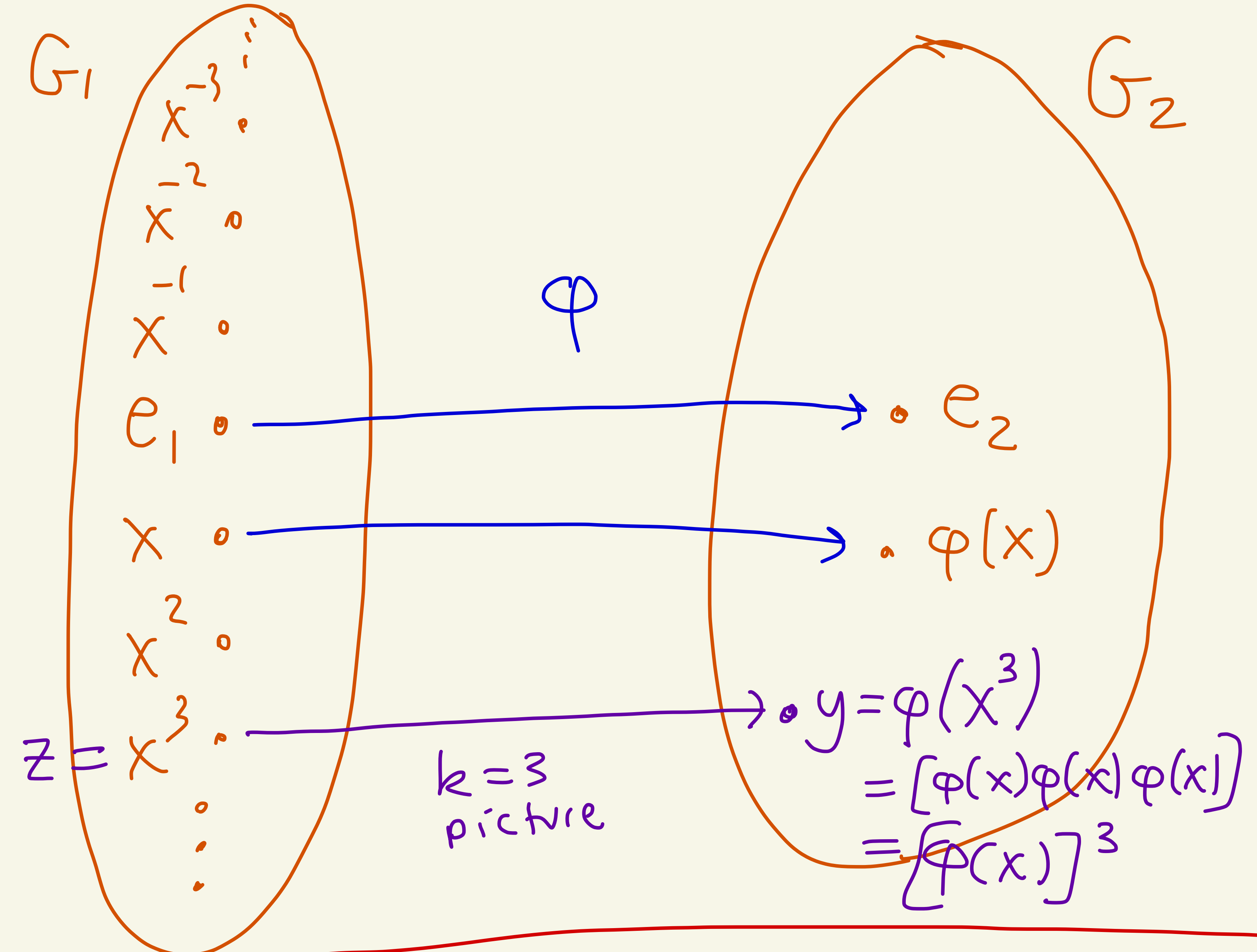
If G_1 is cyclic and φ is onto,

then G_2 is cyclic.

<u>proof:</u>

Since G_1 is cyclic there exists $x \in G_1$ where

$$G_1 = \langle x \rangle = \{ x^k \mid k \in \mathbb{Z} \}$$



Claim: $G_2 = \langle \varphi(x) \rangle$

Let $y \in G_2$.
Since φ is onto there exists $z \in G_1$ where $\varphi(z) = y$.
Since $G_1 = \langle x \rangle$ we know $z = x^k$ where $k \in \mathbb{Z}$

So, $y = \varphi(z) = \varphi(x^k)$

$$= \varphi(x x \cdots x)$$

$$= \varphi(x)\varphi(x)\cdots\varphi(x)$$

$$= [\varphi(x)]^k$$

$\boxed{\text{Claim}}$