Math 4550

11/12/25

HW 4

(4) $\varphi : \mathbb{Z} \to \mathbb{Z}$ is a homomorphism with $\varphi(1) = 5$.

Find $\varphi(4)$, $\varphi(-1)$, $\varphi(-3)$

$$\varphi(4) = \varphi(1+1+1+1)$$
$$= \varphi(1) + \varphi(1) + \varphi(1) + \varphi(1)$$
$$= 5 + 5 + 5 + 5$$
$$= 20$$

$$\boxed{\varphi \text{ is a homomorphism: } \varphi(x^{-1}) = [\varphi(x)]^{-1}}$$

$$\varphi(-1) = [\varphi(1)]^{-1} = [5]^{-1} = -5$$

$$\varphi(-1) = -\varphi(1) = -5$$

$$\varphi(-3) = \varphi((-1)+(-1)+(-1))$$
$$= \varphi(-1) + \varphi(-1) + \varphi(-1)$$
$$= -5 - 5 - 5$$
$$= -15$$

HW 6

(3) $D_8 = \{\cancel{1}, \cancel{r}, \cancel{r^2}, \cancel{r^3}, \cancel{s}, \cancel{sr}, \cancel{sr^2}, \cancel{sr^3}\}$

$$r^4 = 1, \quad s^2 = 1, \quad r^k s = s r^{-k} = s r^{4-k}$$

$$H = \langle r^2 \rangle = \{1, r^2\}$$

You can assume $H \trianglelefteq D_8$. It is.

Let's find the left cosets.

$$H = \{1, r^2\} = r^2 H$$

$$rH = \{r, r^3\} = r^3 H$$

$$sH = \{s, sr^2\} = sr^2 H$$

$$srH = \{sr, sr \cdot r^2\} = \{sr, sr^3\} = sr^3 H$$

$$\boxed{D_8/H = \{H, rH, sH, srH\}}$$

identity

Calculations:

$$(rH)(sH) = (rs)H$$
$$= (sr^{-1})H$$
$$= (sr^{4-1})H$$
$$= (sr^3)H$$
$$= srH$$

orders:

H has order 1

$rH \neq H$

$(rH)(rH) = r^2 H$
$= H$

rH has order 2

$sH \neq H$

$(sH)(sH) = s^2 H = 1 \cdot H = H$

sH has order 2

$srH \neq H$

$$
\begin{aligned}
(srH)(srH) &= (srsr)H \\
&= (ssr^{-1}r)H \\
&= (s^2 \cdot 1)H \\
&= 1 \cdot H \\
&= H
\end{aligned}
$$

so, srH has order 2

D_8/H is <u>not</u> cyclic since no element has order 4.

HW 6

8

Suppose that G is an abelian group and $H \trianglelefteq G$. Prove G/H is abelian.

<u>Proof:</u> Let $x, y \in G/H$.

Then, $x = aH$, $y = bH$ where $a, b \in G$.

Then,

$$xy = (aH)(bH) = (ab)H \underset{\uparrow}{=} (ba)H$$

since G is abelian

$= (bH)(aH) = yx.$

Thus, G/H is abelian

$\blacksquare$

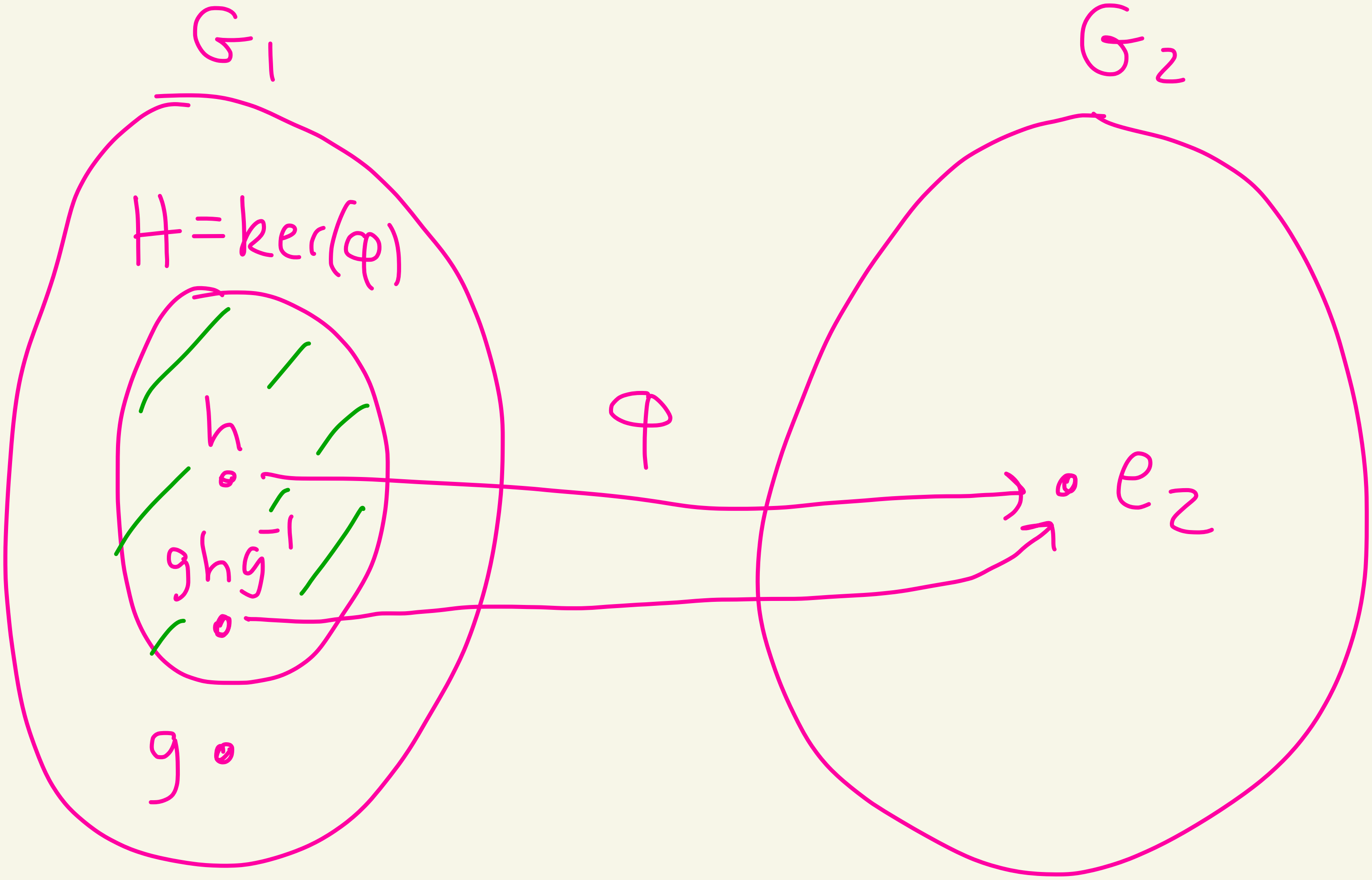
$\boxed{\text{HW 6}}$

Let $\varphi: G_1 \to G_2$ be a homomorphism. Show that $\ker(\varphi)$ is a normal subgroup of G_1.

<u>proof:</u>

By class / HW 4 #6(c) we know $\ker(\varphi)$ is a subgroup of G_1.

Let $H = \ker(\varphi)$.



Let $g \in G_1$ and $h \in H$.
We must show $ghg^{-1} \in H$.
Then,

φ is a homomorphism

$$\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1})$$

$h \in \ker(\varphi)$

$$= \varphi(g)\, e_2\, \varphi(g^{-1})$$

$$= \varphi(g)\varphi(g^{-1})$$

φ is a hom.

$= \varphi(gg^{-1})$

$= \varphi(e_1)$ — e_1 is identity in G_1

$= e_2$ — φ is a hom.

Thus, $ghg^{-1} \in \underbrace{H}_{\ker(\varphi)}$.

So, $H \trianglelefteq G_1$ $\boxtimes$

METHOD 2:

Let $H = \ker(\varphi)$, $g \in G_1$.

We want that $gH = Hg$.

Let's show $gH \subseteq Hg$.

Pick $x \in gH$.

Then, $x = gh$ where $h \in H$.

Then,

$$\varphi(xg^{-1}) = \varphi(ghg^{-1}) = \ldots\ldots = e_2$$

same as above

So, $xg^{-1} \in H$.

Thus, $xg^{-1} = h_1$ where $h_1 \in H$.

So, $x = h_1 g$.

Thus, $x \in Hg$.

So, $gH \subseteq Hg$.

Similarly, $Hg \subseteq gH$. $\blacksquare$