Math 4550

11/19/25

Topic 7 - 1st iso. theorem

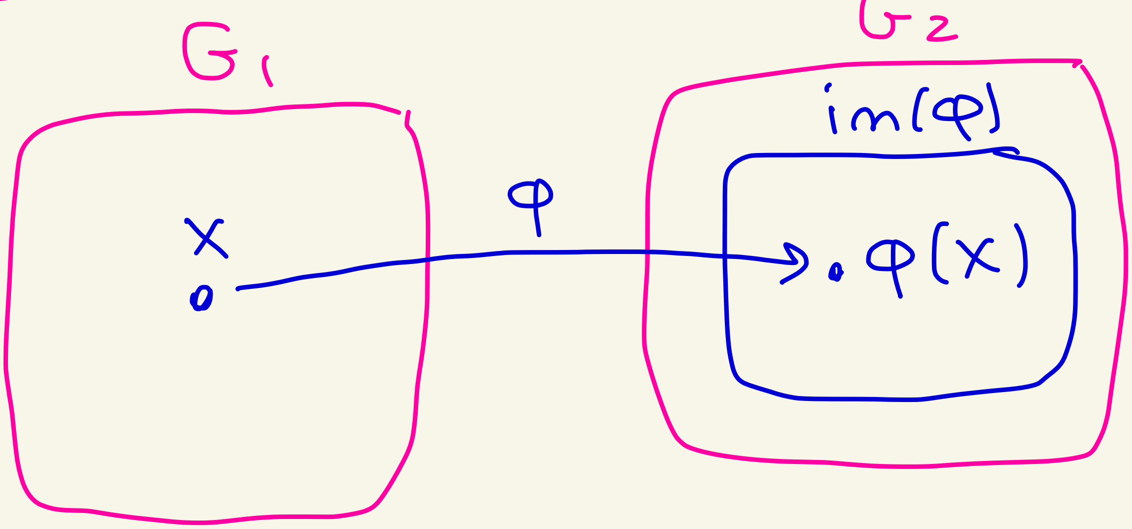
<u>Theorem</u> (First isomorphism theorem)

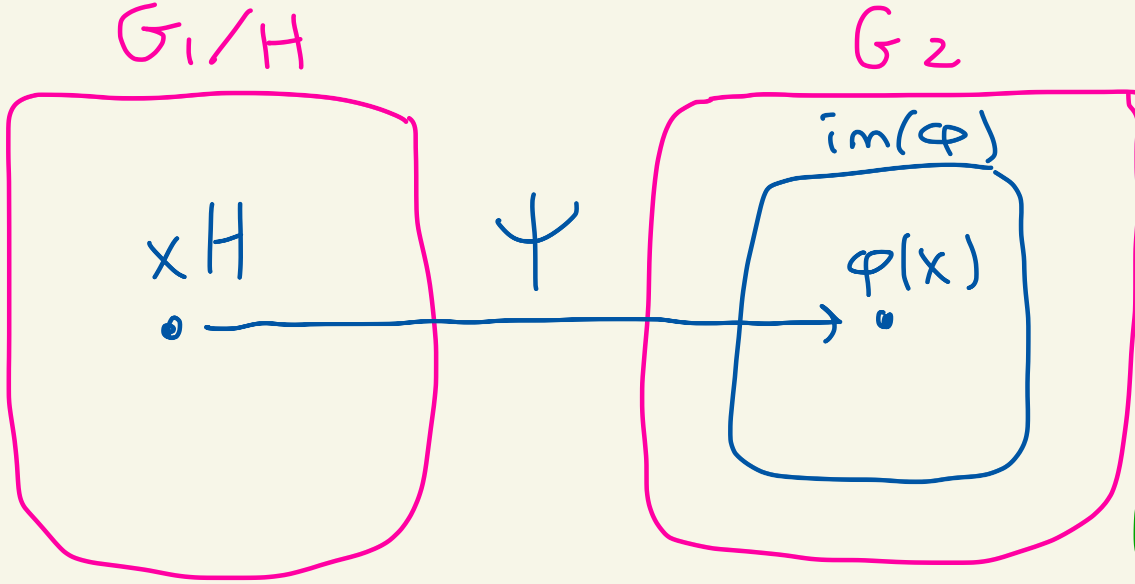
Let $\varphi : G_1 \to G_2$ be a homomorphism where G_1 and G_2 are groups.

Then,

$$G_1 / \ker(\varphi) \cong \operatorname{im}(\varphi)$$

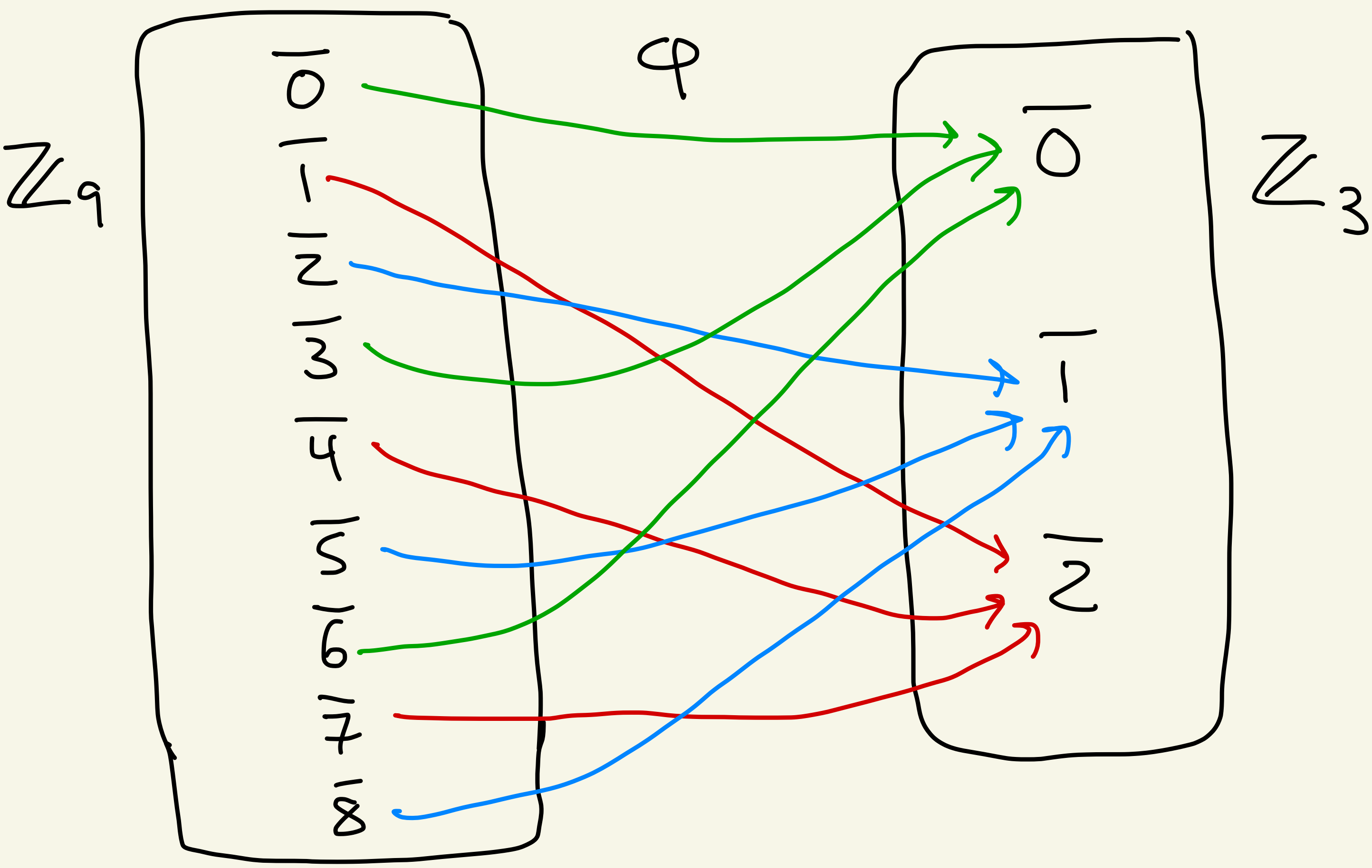
<u>proof outline:</u> Let $H = \ker(\varphi)$





$\psi(xH) = \varphi(x)$ is an isomorphism between G_1/H & $\operatorname{im}(\varphi)$

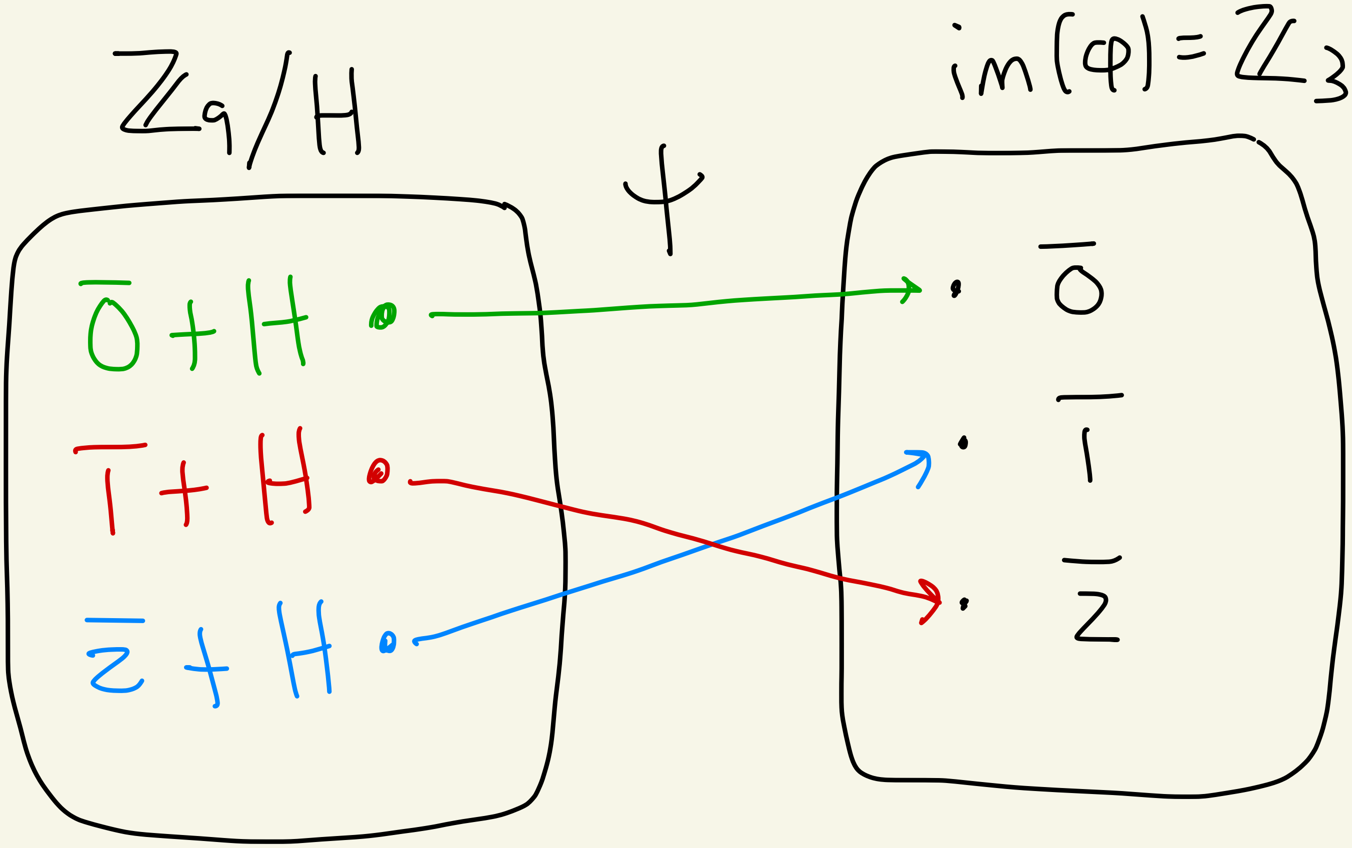
<u>Ex:</u> Define the homomorphism $\varphi : \mathbb{Z}_9 \to \mathbb{Z}_3$ where $\varphi(\overline{1}) = \overline{2}$



$H = \ker(\varphi) = \{\bar{0}, \bar{3}, \bar{6}\}$ ← kernel is always a normal subgroup

$\bar{1} + H = \{\bar{1}, \bar{4}, \bar{7}\}$

$\bar{2} + H = \{\bar{2}, \bar{5}, \bar{8}\}$



Formula:

$$\psi(\bar{a} + H) = \varphi(\bar{a})$$

ψ is an isomurphism

Proof of first isomorphism theorem:

Let $\varphi: G_1 \to G_2$ be a homomorphism.

Let $H = \ker(\varphi)$.

We know from class that $H \trianglelefteq G_1$

So, G_1/H is a group using the operation: $(aH)(bH) = (ab)H$

Define $\psi: G_1/H \to \operatorname{im}(\varphi)$ by $\psi(aH) = \varphi(a)$

Claim 1: ψ is well-defined

pf of claim 1:

Suppose $aH = bH$ where $a, b \in G_1$

We must show: $\psi(aH) = \psi(bH)$

That is we must show $\varphi(a) = \varphi(b)$

Since $aH = bH$ we know $a \in bH$.
So, $a = bh$ where $h \in H$.

Then,

$$
\begin{aligned}
\psi(aH) = \varphi(a) &= \varphi(bh) \\
&= \varphi(b)\varphi(h) \\
&= \varphi(b) \cdot e_2 \\
&= \varphi(b) \\
&= \psi(bH)
\end{aligned}
$$

φ is a hom.

$h \in H$
$H = \ker(\varphi)$

$\boxed{\text{Claim 1}}$

<u>Claim 2: Ψ is an isomorphism</u>

First we show that ψ is a homomorphism.

Let $aH, bH \in G_1/H$.

Then,

$$
\begin{aligned}
\psi((aH)(bH)) &\overset{\text{operation in } G_1/H}{=} \psi((ab)H) \\
&\overset{\text{def of } \psi}{=} \varphi(ab) \\
&\overset{\varphi \text{ is a hom.}}{=} \varphi(a)\varphi(b) \\
&\overset{\text{def of } \psi}{=} \psi(aH)\psi(bH)
\end{aligned}
$$

So, ψ is a homomorphism.

<u>ψ is 1-1:</u>

Suppose $\psi(aH) = \psi(bH)$ where $aH, bH \in G_1/H$.

Then, $\varphi(a) = \varphi(b)$.

So, $\varphi(a)^{-1}\varphi(b) = e_2$ identity in G_2

Then, $\varphi(a^{-1})\varphi(b) = e_2$. φ is a hom.

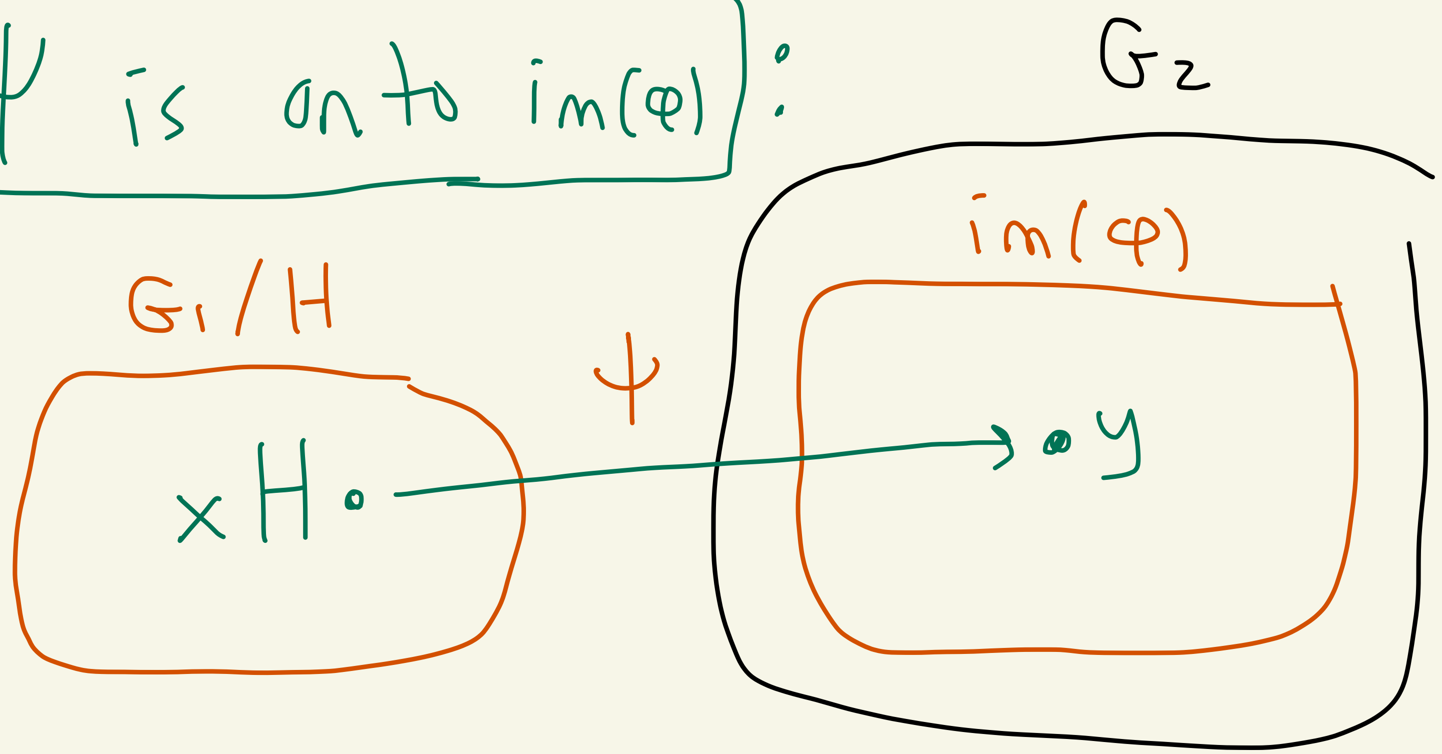
Then, $\varphi(a^{-1}b) = e_2$.

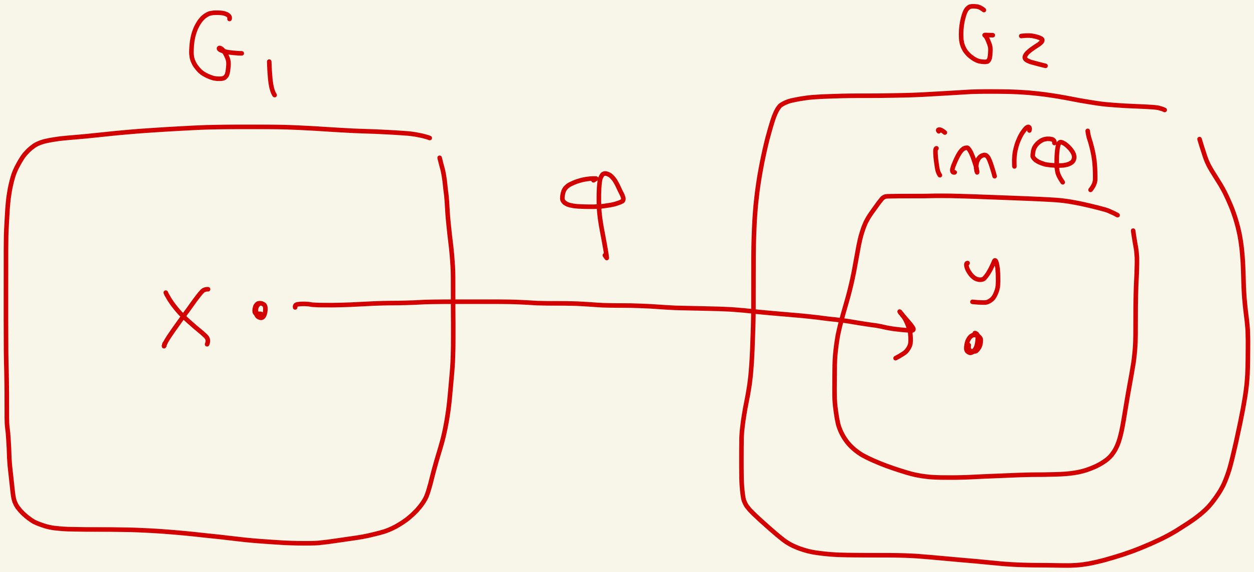
So, $a^{-1}b \in H$ $H = \ker(\varphi)$

By a theorem in class, $aH = bH$.

So, ψ is 1-1.

$\boxed{\psi \text{ is onto } \operatorname{im}(\varphi)}$:





Let $y \in \mathrm{im}(\varphi)$.

Then there exists $x \in G_1$ where $\varphi(x) = y$.

Then, $xH \in G_1/H$ and

$$\psi(xH) = \varphi(x) = y$$

$$\boxed{\text{¡lo lamento!}}$$
