Math 4550

11/3/25

If $H \trianglelefteq G$, then one can turn the set of left cosets into a group using

$$(aH)(bH) = (ab)H$$

We will show this is well-defined.
The identity will be $eH = H$.
The inverse of aH will be $a^{-1}H$

Ex:

$G = \mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

$H = \langle \bar{3} \rangle = \{\bar{0}, \bar{3}\}$

Since $\mathbb{Z}_6$ is abelian, all its subgroups are normal.

So, $H \trianglelefteq \mathbb{Z}_6$.

$H = \{\bar{0}, \bar{3}\}$

<u>left cosets:</u>

$\bar{0} + H = \{\bar{0}, \bar{3}\} = \bar{3} + H$

$\bar{1} + H = \{\bar{1}, \bar{4}\} = \bar{4} + H$

$\bar{2} + H = \{\bar{2}, \bar{5}\} = \bar{5} + H$

$$\underbrace{\mathbb{Z}_6/H}_{\text{set of left cosets}} = \{\underbrace{\bar{0}+H}_{\text{identity element is } \bar{0}+H}, \bar{1}+H, \bar{2}+H\}$$

read: "$\mathbb{Z}_6$ mod H"

group operation for $\mathbb{Z}_6/H$ is:

$$(\bar{a}+H)+(\bar{b}+H) = (\bar{a}+\bar{b})+H$$

Example calculations:

$$
\begin{aligned}
(\bar{2}+H)+(\bar{1}+H) &= (\bar{2}+\bar{1})+H \\
&= \bar{3}+H \\
&= \bar{0}+H \leftarrow \text{identity}
\end{aligned}
$$

So, $\bar{2}+H$ and $\bar{1}+H$ are inverses.

Let's give an example of why this operation is well-defined

$$(\bar{2}+H)+(\bar{1}+H) = \bar{0}+H$$

equal equal

EQUAL

$$
\begin{aligned}
(\bar{5}+H)+(\bar{4}+H) &= (\bar{5}+\bar{4})+H \\
&= \bar{9}+H \\
&= \bar{3}+H = \bar{0}+H
\end{aligned}
$$

in $\mathbb{Z}_6$
$\bar{9}=\bar{3}$

$\boxed{\bar{3}+H=\bar{0}+H}$

Group table for $\mathbb{Z}_6/H$

$\mathbb{Z}_6/H$	$\bar{0}+H$	$\bar{1}+H$	$\bar{2}+H$
$\bar{0}+H$	$\bar{0}+H$	$\bar{1}+H$	$\bar{2}+H$
$\bar{1}+H$	$\bar{1}+H$	$\bar{2}+H$	$\bar{0}+H$
$\bar{2}+H$	$\bar{2}+H$	$\bar{0}+H$	$\bar{1}+H$

$$(\bar{0}+H)+(\bar{2}+H)=(\bar{0}+\bar{2})+H=\bar{2}+H$$

$$(\bar{1}+H)+(\bar{1}+H)=(\bar{1}+\bar{1})+H=\bar{2}+H$$

$$(\bar{2}+H)+(\bar{2}+H)=\bar{4}+H=\bar{1}+H$$

$\mathbb{Z}_6/H = \{\bar{0}+H, \bar{1}+H, \bar{2}+H\}$

is cyclic. Why?

$\bar{1}+H$

$(\bar{1}+H)+(\bar{1}+H) = \bar{2}+H$

$(\bar{1}+H)+(\bar{1}+H)+(\bar{1}+H) = \bar{3}+H = \bar{0}+H$

$\bar{1}+H$ is a generator

Note: $\bar{1}+H$ has order 3 in $\mathbb{Z}_6/H$

$\bar{2}+H$ is also a generator

$\bar{2}+H$

$(\bar{2}+H)+(\bar{2}+H) = \bar{4}+H = \bar{1}+H$

$(\bar{2}+H)+(\bar{2}+H)+(\bar{2}+H) = \bar{6}+H = \bar{0}+H$

Note: $\bar{2}+H$ has order 3 in $\mathbb{Z}_6/H$.

Ex: Consider

identity is $(\bar{0},\bar{0})$

$$G = \mathbb{Z}_4 \times \mathbb{Z}_2 = \{(\bar{0},\bar{0}), (\bar{0},\bar{1}), (\bar{1},\bar{0}), (\bar{1},\bar{1}), (\bar{2},\bar{0}), (\bar{2},\bar{1}), (\bar{3},\bar{0}), (\bar{3},\bar{1})\}$$

$$H = \langle (\bar{2},\bar{0}) \rangle = \{(\bar{0},\bar{0}), (\bar{2},\bar{0})\}.$$

Since $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ is abelian we know $H \trianglelefteq \mathbb{Z}_4 \times \mathbb{Z}_2$.

left cosets:

$$(\bar{0},\bar{0}) + H = \{(\bar{0},\bar{0}), (\bar{2},\bar{0})\} = (\bar{2},\bar{0}) + H$$

$$(\bar{0},\bar{1}) + H = \{(\bar{0},\bar{1}), (\bar{2},\bar{1})\} = (\bar{2},\bar{1}) + H$$

$$(\bar{1},\bar{0}) + H = \{(\bar{1},\bar{0}), (\bar{3},\bar{0})\} = (\bar{3},\bar{0}) + H$$

$$(\bar{1},\bar{1}) + H = \{(\bar{1},\bar{1}), (\bar{3},\bar{1})\} = (\bar{3},\bar{1}) + H$$

So,

$$\mathbb{Z}_4 \times \mathbb{Z}_2 / H = \left\{ \boxed{(\bar{0},\bar{0})+H}, (\bar{0},\bar{1})+H, (\bar{1},\bar{0})+H, (\bar{1},\bar{1})+H \right\}$$

identity element

What's the order of $(\bar{0},\bar{1})+H$?

$(\bar{0},\bar{1})+H \neq (\bar{0},\bar{0})+H$

$$[(\bar{0},\bar{1})+H]+[(\bar{0},\bar{1})+H] = [(\bar{0},\bar{1})+(\bar{0},\bar{1})]+H$$
$$= (\bar{0},\bar{2})+H$$
$$= \boxed{(\bar{0},\bar{0})+H}$$

identity

So, $(\bar{0},\bar{1})+H$ has order 2.

What's the order of $(\bar{1},\bar{0})+H$?

$(\bar{1},\bar{0})+H \neq (\bar{0},\bar{0})+H$

$$[(\bar{1},\bar{0})+H]+[(\bar{1},\bar{0})+H] = [(\bar{1},\bar{0})+(\bar{1},\bar{0})]+H$$
$$= (\bar{2},\bar{0})+H$$

$= (\bar{0},\bar{0}) + H$

$\boxed{\text{from above}}$

So, $(\bar{1},\bar{0})+H$ has order 2.

element	order in $\mathbb{Z}_4 \times \mathbb{Z}_2 / H$
$(\bar{0},\bar{0})+H$	1
$(\bar{1},\bar{0})+H$	2
$(\bar{0},\bar{1})+H$	2
$(\bar{1},\bar{1})+H$	2

Is $\mathbb{Z}_4 \times \mathbb{Z}_2 / H$ cyclic? No.

$|\mathbb{Z}_4 \times \mathbb{Z}_2 / H| = 4$ and no element has order 4

$\mathbb{Z}_4 \times \mathbb{Z}_2 / H$ is not cyclic, but it is abelian.

<u>HW:</u> If G is abelian and $H \trianglelefteq G$, then G/H is abelian.

<u>proof:</u>

Let $a, b \in G$.

Then,

G is abelian

$$(aH)(bH) \underset{\text{def}}{=} (ab)H = (ba)H$$
$$= (bH)(aH)$$

$\blacksquare$