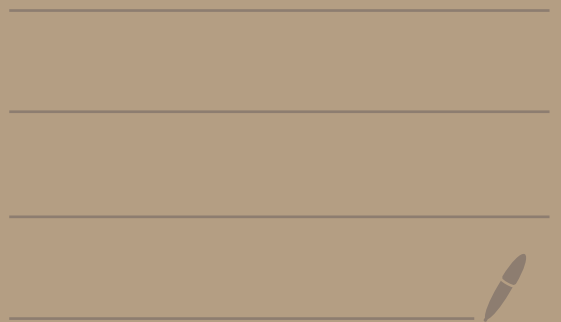


Math 4550

11/5/25



11/5 Topic 6

11/10
Review

11/12
Review

11/17
Test 2

Theorem: Let G be a group and $H \leq G$.

The operation

$$(aH)(bH) = (ab)H$$

on the left cosets is well-defined if and only if H is a normal subgroup.

proof:

($\Leftarrow$) Suppose $H \trianglelefteq G$.

Let's show the above operation on G/H is well-defined.

set of left cosets

Suppose $aH = cH$

$$\text{and } bH = dH$$

where $a, b, c, d \in G$.

We must show that

$$(aH)(bH) = (ab)H$$

is equal to

$$(cH)(dH) = (cd)H$$

Since $aH = cH$ and $a \in aH$
 $a = ae$

we know $a \in cH$.

So, $a = ch_1$ where $h_1 \in H$.

Similarity since $bH = dH$ we
have $b = dh_2$ where $h_2 \in H$.

Then,

$$ab = (ch_1)(dh_2) = c\underline{h_1 d}h_2$$

Since $H \trianglelefteq G$ we know

$$Hd = dH.$$

Since $h, d \in Hd$ we

know $h, d \in dH$.

So, $h, d = dh_3$ where $h_3 \in H$.

Then,

$$ab = c\underline{h_1 d}h_2 = c\underline{d}h_3h_2$$

But $h_3h_2 \in H$ because $H \leq G$,

so $ab = cdh_3h_2 \in (cd)H$.

Thus, $(ab)H = (cd)H$

recall:
 $xH = yH$
iff $x \in yH$

($\Rightarrow$) See online notes.



Theorem: Let G be a group and $H \trianglelefteq G$.

Then, G/H $\leftarrow$ set of left cosets,

is a group under the operation $(aH)(bH) = (ab)H$.

The identity is $eH = H$

The inverse of aH is $a^{-1}H$.

e is the identity of G

proof:

① (closure) Let $x, y \in G/H$.

Then, $x = aH$, $y = bH$

where $a, b \in G$.

We know $ab \in G$ because

G is a group.

So,

$$xy = (aH)(bH) = (ab)H \in G/H$$

def

② Let $x, y, z \in G/H$.

Then, $x = aH$, $y = bH$, $z = cH$

where $a, b, c \in G$.

We get

$$(aH)[(bH)(cH)] = (aH)[(bc)H]$$

$$= a(bc)H$$

$$= (ab)cH$$

$$= [(ab)H](cH)$$

$$= [(aH)(bH)](cH)$$

G is
associative

def

③ Given $aH \in G/H$ we have

$$(aH)(eH) = (ae)H = aH$$

$$(eH)(aH) = (ea)H = aH$$

So, eH is an identity element for G/H where e is the identity for G .

④ Let $aH \in G/H$ where $a \in G$.

Since G is a group, a^{-1} exists in G .

We get

$$(aH)(a^{-1}H) = (aa^{-1})H = eH$$

$$(a^{-1}H)(aH) = (a^{-1}a)H = eH$$

Thus, $a^{-1}H$ is an inverse of aH in G/H .



Ex:

$$G = D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$$

$$H = \langle s \rangle = \{1, s\}$$

$$s^2 = 1$$

$$r^4 = 1$$

$$r^k s = s r^{-k} = s r^{4-k}$$

Claim: $H \triangleleft D_8$

Proof:

left cosets:

$$H = \{1, s\} = sH$$

$$rH = \{r, rs\} = \{r, sr^{-1}\} = \{r, sr^3\} = (sr^3)H$$

$$r^2H = \{r^2, r^2s\} = \{r^2, sr^{-2}\} = \{r^2, sr^2\} = (sr^2)H$$

$$r^3H = \{r^3, r^3s\} = \{r^3, sr\} = (sr)H$$

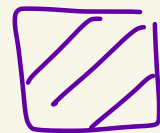
right cosets:

$$H = \{1, s\} = Hs$$

$$Hr = \{r, sr\} = H(sr)$$

$$Hr^2 = \{r^2, sr^2\} = H(sr^2)$$

$$Hr^3 = \{r^3, sr^3\} = H(sr^3)$$



Ex: $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

$$H = \{1, r, r^2, r^3\}$$

left cosets

$$H = \{1, r, r^2, r^3\}$$

$$sH = \{s, sr, sr^2, sr^3\}$$

right cosets

$$H = \{1, r, r^2, r^3\}$$

$$Hs = \{s, rs, r^2s, r^3s\} = \{s, sr^{-1}, sr^{-2}, sr^{-3}\}$$

$$= \{s, sr^3, sr^2, sr\}$$

$$H \trianglelefteq D_8$$

$$D_8/H = \{H, sH\}$$

identity

D_8/H	H	sH
H	H	sH
sH	sH	H

$$(sH)(sH) = s^2H = 1H = H$$

order of H is 1

order of sH is 2

D_8/H is cyclic, generated by sH