Math 4550

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Theorem: Let G be a group and $H \leq G$.

The operation

$$(aH)(bH) = (ab)H$$

on the left cosets is well-defined if and only if H is a normal subgroup.

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proof:

($\Leftarrow$) Suppose $H \trianglelefteq G$.

Let's show the above operation on G/H is well-defined.

set of left cosets

Suppose $aH = cH$

and $bH = dH$

where $a, b, c, d \in G$.

We must show that

$$(aH)(bH) = (ab)H$$

is equal to

$$(cH)(dH) = (cd)H$$

Since $aH = cH$ and $\underbrace{a \in aH}_{a = ae}$

we Know $a \in cH$.

So, $a = ch_1$ where $h_1 \in H$.

Similary since $bH = dH$ we have $b = dh_2$ where $h_2 \in H$.

Then,

$$ab = (ch_1)(dh_2) = ch_1 dh_2$$

Since $H \trianglelefteq G$ we know $Hd = dH$.

Since $h_1 d \in Hd$ we know $h_1 d \in dH$.

So, $h_1 d = dh_3$ where $h_3 \in H$.

Then,

$$ab = ch_1 dh_2 = c dh_3 h_2$$

But $h_3 h_2 \in H$ because $H \leq G$,

so $ab = cd h_3 h_2 \in (cd)H$.

Thus, $(ab)H = (cd)H$ ⟵ $\boxed{\text{recall: } xH = yH \text{ iff } x \in yH}$

($\Rightarrow$) See online notes. $\blacksquare$

<u>Theorem:</u> Let G be a group and $H \trianglelefteq G$.

Then, G/H $\leftarrow$ set of left cosets

is a group under the operation $(aH)(bH) = (ab)H$.

The identity is $eH = H$

The inverse of aH is $a^{-1}H$.

e is the identity of G

proof:

① (closure) Let $x, y \in G/H$.

Then, $x = aH$, $y = bH$ where $a, b \in G$.

We know $ab \in G$ because G is a group.

So,

$$xy = (aH)(bH) \overset{\text{def}}{=} (ab)H \in G/H$$

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② Let $x, y, z \in G/H$.

Then, $x = aH$, $y = bH$, $z = cH$ where $a, b, c \in G$.

We get

$$
\begin{aligned}
(aH)[(bH)(cH)] &\overset{\text{def}}{=} (aH)[(bc)H] \\
&\overset{\text{def}}{=} a(bc)H \\
&\overset{\text{G is associative}}{=} (ab)cH \\
&\overset{\text{def}}{=} [(ab)H](cH) \\
&\overset{\text{def}}{=} [(aH)(bH)](cH)
\end{aligned}
$$

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3 Given $aH \in G/H$ we have

$$(aH)(eH) = (ae)H = aH$$

$$(eH)(aH) = (ea)H = aH$$

So, eH is an identity element for G/H where e is the identity for G.

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④ Let $aH \in G/H$ where $a \in G$.

Since G is a group, a^{-1} exists in G.

We get

$$(aH)(a^{-1}H) = (aa^{-1})H = \overbrace{eH}^{\text{identity of } G/H}$$

$$(a^{-1}H)(aH) = (a^{-1}a)H = eH$$

Thus, $a^{-1}H$ is an inverse of aH in G/H. $\blacksquare$

Ex:

$G = D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

$H = \langle s \rangle = \{1, s\}$

$$\boxed{\begin{aligned} s^2 &= 1 \\ r^4 &= 1 \\ r^k s &= s r^{-k} \\ &= s r^{4-k} \end{aligned}}$$

Claim: $H \not\trianglelefteq D_8$

proof:

left cosets:

$H = \{1, s\} = sH$

$rH = \{r, rs\} = \{r, sr^{-1}\} = \{r, sr^3\} = (sr^3)H$

$r^2H = \{r^2, r^2s\} = \{r^2, sr^{-2}\} = \{r^2, sr^2\} = (sr^2)H$

$r^3H = \{r^3, r^3s\} = \{r^3, sr\} = (sr)H$

right cosets:

$H = \{1, s\} = Hs$

$Hr = \{r, sr\} = H(sr)$

$Hr^2 = \{r^2, sr^2\} = H(sr^2)$

$Hr^3 = \{r^3, sr^3\} = H(sr^3)$ $\blacksquare$

<u>Ex:</u> $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

$H = \{1, r, r^2, r^3\}$

<u>left cosets</u>

$H = \{1, r, r^2, r^3\}$

$sH = \{s, sr, sr^2, sr^3\}$

<u>right cosets</u>

$H = \{1, r, r^2, r^3\}$

$Hs = \{s, rs, r^2s, r^3s\} = \{s, sr^{-1}, sr^{-2}, sr^{-3}\}$

$= \{s, sr^3, sr^2, sr\}$

$H \trianglelefteq D_8$

identity

$D_8/H = \{ H, sH \}$

D_8/H	H	sH
H	H	sH
sH	sH	H

$$\boxed{(sH)(sH) = s^2H = 1H = H}$$

order of H is 1

order of sH is 2

D_8/H is cyclic, generated by sH