III. 13 – Homomorphisms and Factor Groups

Def: A map ϕ of a group G into a group G' is a homomorphism if

$$\phi(ab) = \phi(a)\phi(b)$$

for all $a, b \in G$.

Prop: Let $\phi: G \to G'$ be a group homomorphism of G onto G'. If G is abelian, then G' is abelian.

pf: Let $a', b' \in G'$ and $a, b \in G$ s.t. $\phi(a) = a'$, $\phi(b) = b'$.
Then $a'b' = \phi(a)\phi(b) = \phi(ab) \underset{\uparrow}{=} \phi(ba) = \phi(b)\phi(a) = b'a'$ $\square$
G abelian

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ex: Let $GL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R},\ ad - bc \neq 0 \right\}$

Then, $\det: GL(2,\mathbb{R}) \longrightarrow \mathbb{R}^* = \mathbb{R} \setminus \{0\}$

is a homomorphism.

<u>ex:</u> Let $G = G_1 \times G_2 \times \dots \times G_n$ where G_i is a group. The <u>projection map</u> $\pi_i : G \to G_i$ where $\pi_i(g_1, g_2, \dots, g_i, \dots, g_n) = g_i$ is a homomorphism for $i = 1, \dots, n$.

For example,

$$\pi_1 : \cancel{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}} \qquad \mathbb{Z}_2 \times \mathbb{Z}_3 \longrightarrow \mathbb{Z}_2$$

$$(a, b) \longmapsto a$$

$$\pi_2 : \cancel{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{Z}/3\mathbb{Z}} \qquad \mathbb{Z}_2 \times \mathbb{Z}_3 \longrightarrow \mathbb{Z}_3$$

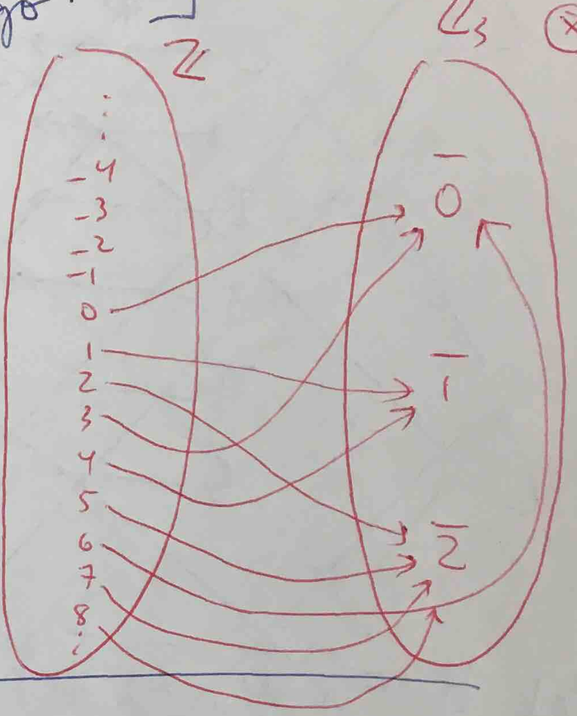
$$(a, b) \longmapsto b$$

are homomorphisms

(reduction modulo n)

ex: Let $\phi : \mathbb{Z} \to \cancel{\mathbb{Z}/n\mathbb{Z}}\ \mathbb{Z}_n$ be the map $\phi(n) = \overline{n}$. Then ϕ is a homomorphism.

$$\left[\ \phi(n+m) = \underbrace{\overline{n+m} = \overline{n} + \overline{m}}_{\text{we showed this a long time ago}} = \phi(n) + \phi(m)\ \right]$$



For example,

$$\phi : \mathbb{Z} \to \cancel{\mathbb{Z}/3\mathbb{Z}}\ \mathbb{Z}_3$$

$$6 \longmapsto \overline{6} = \overline{0}$$

$$1 \longmapsto \overline{1}$$

$$7 \longmapsto \overline{1}$$

Def: Let ϕ be a mapping of a set X onto a set Y, and let $A \subseteq X$ and $B \subseteq Y$. The image $\phi[A]$ of A in Y under ϕ is $\{\phi(a) \mid a \in A\}$. The set $\phi[X]$ is the range of ϕ. The inverse image $\phi^{-1}[B]$ of B in X is $\{x \in X \mid \phi(x) \in B\}$

Ex: Use the example above and do

$$\varphi[\mathbb{Z}] = \mathbb{Z}_3,\quad \varphi[3\mathbb{Z}] = \{\overline{0}\},\quad \varphi^{-1}[\{\overline{0}\}] = 3\mathbb{Z}.$$

11/7 Integers are cyclic under addition

Monday Week 12
Nov. 7, 2016

Nov 7 Notes

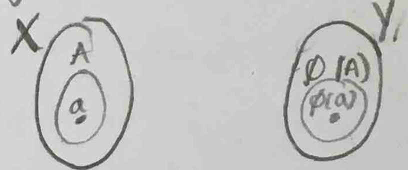
$\langle 1 \rangle = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$1 \quad 1+1 \quad 1+1+1$

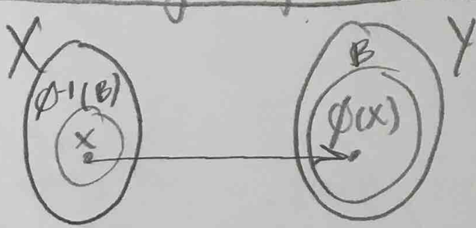
1 is a generator so $\mathbb{Z}$ is cyclic.

Def: Let X and Y be sets and $\phi: X \to Y$

- Let $A \subseteq X$, then the <u>image of A under ϕ</u> is defined to be $\phi(A) = \{\phi(a) \mid a \in A\}$



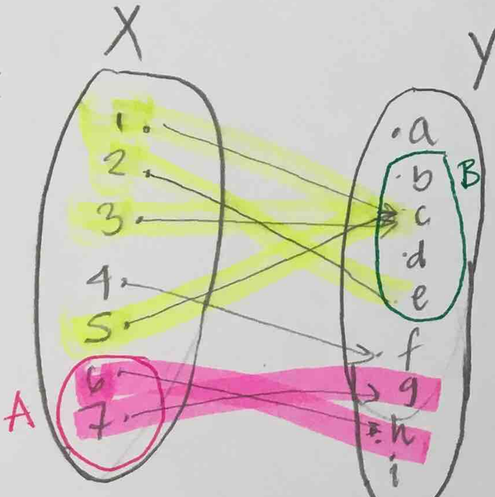
- Let $B \subseteq Y$. Then the <u>inverse image of B under ϕ</u> is $\phi^{-1}(B) = \{x \mid \phi(x) \in B\}$



Ex:

$\phi^{-1}(B)$

$\phi(A)$ A



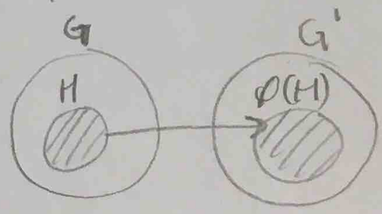
$\phi^{-1}(B) = \{1, 2, 3, 5\}$

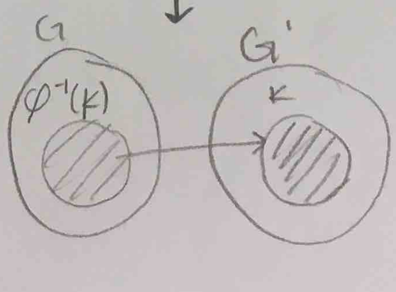
$\phi(A) = \{\phi(6), \phi(7)\} = \{g, h\}$

Theorem: Let $\phi: G \to G'$ be a homomorphism

(a) Let $H \leq G$, then $\phi(H) \leq G'$

(b) Let $K \leq G'$ then $\phi^{-1}(K) \leq G$



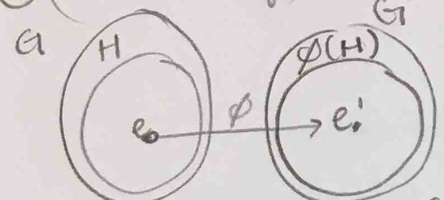


Proof (a):

Let e and e' be the identity elements of G and G'
let $H \leq G$

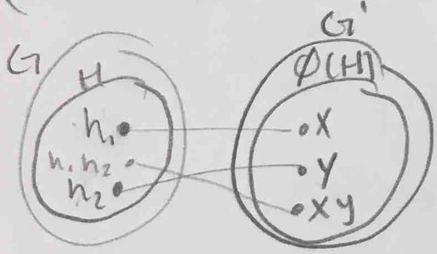
Remember
$\phi(H) = \{\phi(h) \mid h \in H\}$

① (identity)



we know that $e \in H$ since H is a subgroup of G
so $\phi(e)$ is in $\phi(H)$
since ϕ is a homomorphism we know $\phi(e) = e'$
so $e' \in \phi(H)$

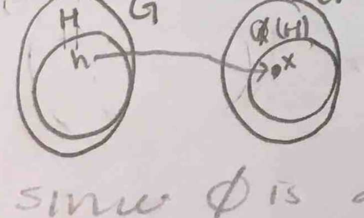
(closure)



let $x, y \in \phi(H)$
we want to show that
$xy \in \phi(H)$.

since $x, y \in \phi(H)$, $\exists\ h_1, h_2 \in H$ where
$\phi(x) = h_1$ and $\phi(y) = h_2$
since H is a subgroup, $h_1 h_2 \in H$
since ϕ is a homomorphism
$\phi(h_1 h_2) = \phi(h_1)\phi(h_2) = xy$ ✓ so $xy \in \phi(H)$

(inverses)



let $x \in \phi(H)$ we want to show that
$x^{-1} \in \phi(H)$, since $x \in \phi(H)$ then $\exists\ h \in H$
s.t. $\phi(h) = x$. since $H \leq G$ and $h \in H$
we know that $h^{-1} \in H$
since ϕ is a homomorphism $\phi(h^{-1}) = [\phi(h)]^{-1} = x^{-1}$

so $x^{-1} \in \phi(H)$