11/21 P1 Week 11 Monday November 21, 2016

Recall: $H \trianglelefteq G$ (normal subgroup) $\} \; gH = Hg \;\; \forall g \in G$

- G/H is the set of left cosets

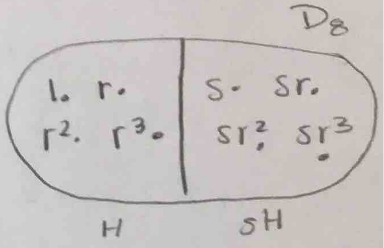
  G/H is a group under $(aH)(bH) = (ab)H$

Example: $G = D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

left cosets:

$H = \langle r \rangle = \{1, r, r^2, r^3\} = rH = r^2H = r^3H$

$sH = \{s, sr, sr^2, sr^3\} = (sr)H = (sr^2)H = (sr^3)H$



Right cosets:

$H = \{1, r, r^2, r^3\} = Hr = Hr^2 = Hr^3$

$Hs = \{s, rs, r^2s, r^3s\} = H(sr) = H(sr^2) = H(sr^3)$

- Here the right and left cosets are the same so H is normal in D_8.

- so, $D_8/H = \{H, sH\}$ is a group
  ↳ identity element

$(H)\cdot(sH) = (1\cdot H)(s\cdot H) = (1\cdot s)H = sH$

$(sH)(sH) = (s\cdot s)H = 1\cdot H = H$

equal

$(H)(sH) = (rH)(sH)$

$= (rs)H = (sr^3)H$

$= sH$

Table:

D_8/H	H	sH
H	H	sH
sH	sH	H

Fact: Let G be an abelian group and H be a subgroup of G, Then H is normal

proof: Let $g \in G$, Then

$gH = \{gh \mid h \in H\} \underset{\uparrow}{=} \{hg \mid h \in H\} = Hg$ ▨

G is abelian

Fact: Let G be a group

Let H be a subgroup. If there are only 2 left cosets of H, then H is normal.

Example:

$G = \mathbb{Z} = \{\dots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots\}$

$H = 4\mathbb{Z} = \{\dots -8, -4, 0, 4, 8, \dots\}$

$1 + H = \{\dots -7, -3, 1, 5, 9, \dots\}$

$2 + H = \{\dots, -6, -2, 2, 6, 10, \dots\}$

$3 + H = \{\dots, -5, -1, 3, 7, 11, \dots\}$

$\mathbb{Z}$

...	-8	-4	0	4	8	...	$H = 0 + H$
...	-7	-3	1	5	9	...	$1 + H$
...	-6	-2	2	6	10	...	$2 + H$
...	-5	-1	3	7	11	...	$3 + H$

- since $\mathbb{Z}$ is abelian, H is normal so, $\mathbb{Z}/4\mathbb{Z}$ you get a group

$\mathbb{Z}/\mathbb{Z}4 = \{0+H, 1+H, 2+H, 3+H\}$

$\uparrow$ identity

$(3+H) + (2+H) = (3+2) + H = 5 + H = 1 + H$

- what is the order of $2 + H$ in $\mathbb{Z}/4\mathbb{Z}$?

$2 + H \neq 0 + H$

$(2+H) + (2+H) = 4 + H = 0 + H \leftarrow$ identity

so $2+H$ has order 2.

- whats the order of $3+H$

$3 + H \neq 0 + H$

$(3+H) + (3+H) = 6 + H = 2 + H \neq 0 + H$

$(3+H) + (3+H) + (3+H) + (3+H) = (12 + H) = 0 + H \leftarrow$ identity

so $(3+H)$ has order 4 so, $3+H$ is a generator of $\mathbb{Z}/4\mathbb{Z}$ so $\mathbb{Z}/4\mathbb{Z}$ is cyclic.

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Example:

$G = \mathbb{Z}_4 \times \mathbb{Z}_4 = \{(\bar{0},\bar{0}), (\bar{0},\bar{1}), (\bar{0},\bar{2}), (\bar{0},\bar{3}), (\bar{1},\bar{0}), (\bar{1},\bar{1}), (\bar{1},\bar{2}), (\bar{1},\bar{3}),$
$(\bar{2},\bar{0}), (\bar{2},\bar{1}), (\bar{2},\bar{2}), (\bar{2},\bar{3}), (\bar{3},\bar{0}), (\bar{3},\bar{1}), (\bar{3},\bar{2}), (\bar{3},\bar{3})\}$

$H = \langle(\bar{1},\bar{1})\rangle = \{(\bar{0},\bar{0}), (\bar{1},\bar{1}), (\bar{2},\bar{2}), (\bar{3},\bar{3})\} = (\bar{0},\bar{0}) = H$

$(\bar{0},\bar{1}) + H = \{(\bar{0},\bar{1}), (\bar{1},\bar{2}), (\bar{2},\bar{3}), (\bar{3},\bar{0})\}$

$(\bar{0},\bar{2}) + H = \{(\bar{0},\bar{2}), (\bar{1},\bar{3}), (\bar{2},\bar{0}), (\bar{3},\bar{1})\}$

$(\bar{0},\bar{3}) + H = \{(\bar{0},\bar{3}), (\bar{1},\bar{0}), (\bar{2},\bar{1}), (\bar{3},\bar{2})\}$

If G and H are abelian then $G \times H$ is abelian

we know $\mathbb{Z}_4 \times \mathbb{Z}_4$ is abelian
so, H is normal.

so $\mathbb{Z}_4 \times \mathbb{Z}_4/H$ is a group

$\mathbb{Z}_4 \times \mathbb{Z}_4/H = \{(\bar{0},\bar{0}) + H, (\bar{0},\bar{1}) + H, (\bar{0},\bar{2}) + H, (\bar{0},\bar{3}) + H\}$

identity

order of $(\bar{0},\bar{1}) + H$

$[(\bar{0},\bar{1}) + H] + [(\bar{0},\bar{1}) + H] + [(\bar{0},\bar{1}) + H] + [(\bar{0},\bar{1}) + H] = (\bar{0},\bar{0}) + H \leftarrow$ identity

so $(\bar{0},\bar{1}) + H$ has order 4

Thus $\underbrace{\mathbb{Z}_4 \times \mathbb{Z}_4/H}_{\text{cyclic}} = \langle(\bar{0},\bar{1}) + H\rangle$

Theorem: Let G be a cyclic group and Let H be a subgroup. Then G/H is a cyclic group.

proof:

since G is cyclic, we know G is abelian,

so H is normal so G/H is a group.

Since G is cyclic, we know $G = \langle x \rangle$ where $x \in G$.

claim, $G/H = \langle xH \rangle$

Let $g \in G$

Then $g = x^k$ where $k \in \mathbb{Z}$

so, $(gH) = x^k H = (xH)^k$

Ex: $x^3H = (xH)(xH)(xH)$

$x^{-2}H = (x^{-1}H)(x^{-1}H) = (xH)^{-1}(xH)^{-1} = (xH)^{-2}$

so, $G/H = \langle xH \rangle$ $\boxtimes$

Try proving

Let G be a abelian group

Let H be a subgroup. then G/H

is a abelian group