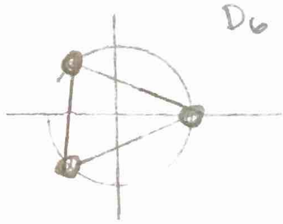
Monday Week 4 Sept. 12, 2016

Dihedral Groups

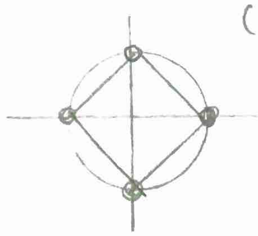
For each $n \geq 3$, let D_{2n} be the set of symmetries of a regular n-gon, where a symmetry is any rigid motion of the n-gon which can be effected by taking a copy of the n-gon, moving this copy in any fashion in 3d-space and then placing the copy back on the original n-gon so it exactly covers it.

- given two symmetries $\sigma_1, \sigma_2 \in D_{2n}$ the group operation $\sigma_1 * \sigma_2$ (written $r_1 r_2$) means first apply r_2 and then apply σ_1. The identity symmetry fixes the n-gon. The inverse σ^{-1} of r is the symmetry that "undoes" σ.

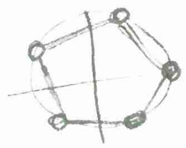
3-gon D_6



4-gon (D_8)

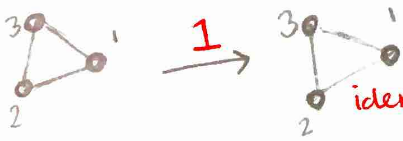


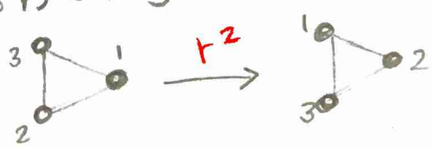
$n=5$ (D_{10})



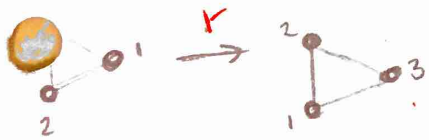
Example

D_6 $(n=3)$ $D_6 = \{1, r, r^2, s, sr, sr^2\}$



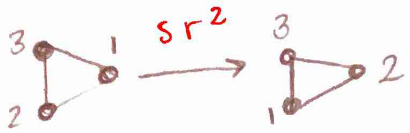


counter clockwise rotation or 2-clockwise rotation.



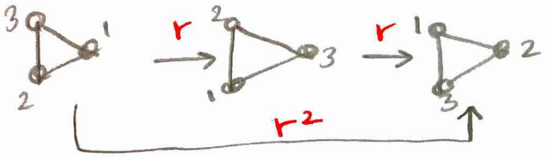
(rotate it once clockwise)

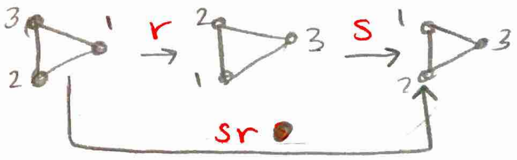
reflect on the x-axis

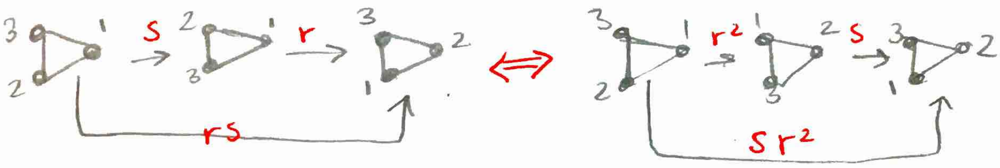


rotate counterclockwise and flip.

* $\sigma_1 \sigma_2$
← read that way







$rs = sr^2$

Side note

$r^3 = 1$

$r^2 r = 1$

$r^{-1} = r^2$

$s^2 = 1$

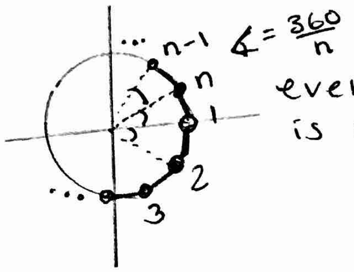
$s \cdot s = 1$

$s^{-1} = s$

9/12 P.2

D_{2n} in general

Fix an n-gon centered at the origin in the xy-plane and label the vertices consecutively from 1 to n in a clockwise manner.



every angle is $= 360°/n$

Let r be the rotation of the n-gon in a clockwise direction by $\frac{360°}{n}$ and let s be the reflection of the n-gon across the x-axis and let 1 be the identity symmetry.

Then:

(1) $1, r, r^2, \ldots, r^{n-1}$ are distinct elements and $r^n = 1$, so $r^{-1} = r^{n-1}$

(2) $s^2 = 1$, so $s^{-1} = s$

(3) $s \neq r^i$ for any i

(4) $sr \neq sr^j$ if $0 \leq i < j \leq n-1$

(5) $D_{2n} = \{1, r, r^2, \ldots r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}$

(6) $rs = sr^{-1}$

(7) $r^i s = sr^{-i}$ for any i

(8) $r^{-i} = r^{n-i}$ for any i

(n=3)

D_6	1	r	r^2	s	sr	sr^2
1	1	r	r^2	s	sr	sr^2
r	r	r^2	1	sr^2	s	sr
r^2	r^2	1	r	sr	sr^2	s
s	s	sr	sr^2	1	r	r^2
sr	sr	sr^2	s	r^2	1	r
sr^2	sr^2	s	sr	r	r^2	1

$(r)(r) = r^2$

$(sr)(sr^2) = s\underline{rs}r^2$
$= s\underline{sr^{-1}}r^2$
$s^2r = 1 \cdot r = r$

$(r)(s) = sr^{-1} = sr^2$

$s(sr) = s^2r = 1 \cdot r = r$

calculate

$r^3 s r^4 s s r^{-2} s r^2$ in D_{10}

$(sr)(sr) = sr\,sr = ssr^{-1}r$
$s^2 = 1$

simplify. $sr^?$

9/14 P.1 Wednesday Week 4 Sept. 14, 2016

Calculate $r^3 s r^4 s s r^{-2} s r^2$ in D_{10}

$D_{10} = \{1, r, r^2, r^3, r^4, s, sr, sr^2, sr^3, sr^4\}$

$D_{2(5)}$ $\quad r^n = r^5 = 1$
$n=5$ $\quad s^2 = 1$ ★

$rs = sr^{-1} = sr^4$

$\boxed{r^{n-1} = r^{-1}}$ $\quad r^4 = r^{-1}$

★ $r^i s = sr^{-i} = sr^{n-i} = sr^{5-i}$

$r^3 s r^4 s s r^{-2} s r^2$

$= r^3 s r^2 s r^2$

$= r^3 \underbrace{ss}_{1} \underbrace{r^{-2} r^2}_{1}$

$= r^3 \cdot 1 \cdot 1 = r^3$

In D_{12}, calculate $r^{-7} s r^3 s r^5$

$D_{12} = \{1, r, r^2, r^3, r^4, r^5, s, sr, sr^2, sr^3, sr^4, sr^5\}$

$n=6$

$r^6 = 1$

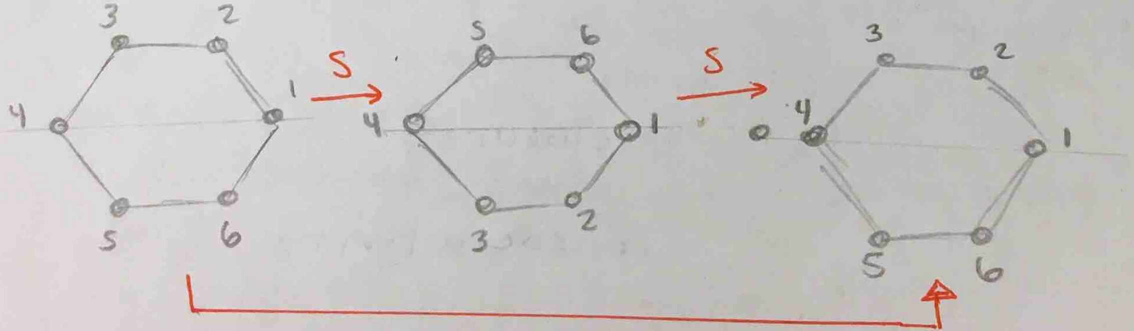
$s^2 = 1$

$r^i s = sr^{-i}$

$r^{-7} s r^3 s r^5 = r^{-7} \underbrace{ss}_{1} \underbrace{r^{-3} r^5}_{r^2}$

$= r^{-7} r^2 = r^{-5} = r^6 r^{-5} = r$

$s^2 = 1$



$ss = s^2$

Def: Let G be a finite group.
The **order** of G is the number of elements in G and is denoted by $|G|$

Example: $|\mathbb{Z}_6| = |\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}| = 6$, $|D_4| = 4$, $|D_{10}| = 10$

HW #2

Subgroup

Def: Let G be a group with operation $*$
Let H be a subset of G, If H itself is a group using the same operation $*$ as G, then we call H a **subgroup** of G and we write $H \leq G$

subset	subgroup
$H \subseteq G$	$H \leq G$

Good test Question ↙

Example: $G = \mathbb{Z}_{12} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \overline{10}, \overline{11}\}$

↑ group under addition

$H = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$

$\boxed{\text{Claim}}$: $H \leq G$ ← "H is a subgroup of G"

$(H,+)$	$\bar{0}$	$\bar{3}$	$\bar{6}$	$\bar{9}$
$\bar{0}$	$\bar{0}$	$\bar{3}$	$\bar{6}$	$\bar{9}$
$\bar{3}$	$\bar{3}$	$\bar{6}$	$\bar{9}$	$\bar{0}$
$\bar{6}$	$\bar{6}$	$\bar{9}$	$\bar{0}$	$\bar{3}$
$\bar{9}$	$\bar{9}$	$\bar{0}$	$\bar{3}$	$\bar{6}$

axioms

① closure : yes, by table every calculation is back in H.

② associativity : we want to know if $\bar{a} + (\bar{b} + \bar{c}) = (\bar{a} + \bar{b}) + \bar{c}$ $\forall\, \bar{a}, \bar{b}, \bar{c}$ in H
since $H \subseteq \mathbb{Z}_{12}$ and $\mathbb{Z}_{12}$ is associative we know that H is too.

③ identity yes, $\bar{0} \in H$

9/14 P. 2

④ Inverse

element	inverse	inverse in H ?
$\overline{0}$	$\overline{0}$	yes
$\overline{3}$	$\overline{9}$	yes
$\overline{6}$	$\overline{6}$	yes
$\overline{9}$	$\overline{3}$	yes

Thus $H \leq \mathbb{Z}_{12}$

☒

Theorem

Let G be a group and H be a subset of G. Then H is a subgroup of G iff

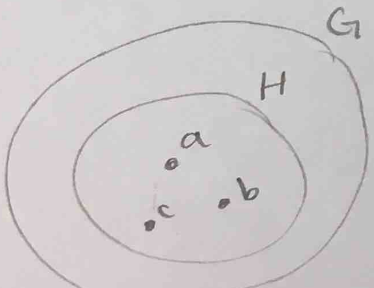
(1) H is closed under the operation of G

(2) The identity of G is in H

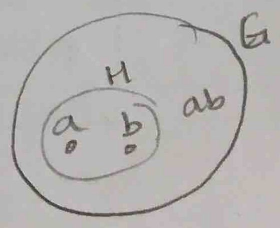
(3) $\forall\ x \in H\ \exists\ x^{-1} \in H$

$\forall\ x, y \in H$ we have $x * y \in H$

why we dont have to check associativity



$a(bc) = (ab)c$

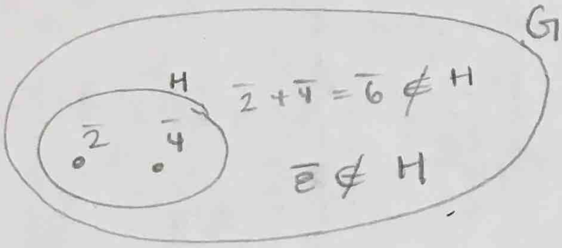


Example:

$G = \mathbb{Z}_{12} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}, \bar{11}\}$

$H = \{\bar{0}, \bar{2}, \bar{4}, \bar{10}\}$

Is H a subgroup of G?



$\boxed{\text{answer}}$

No H isn't closed since $\bar{2}+\bar{4}=\bar{6} \notin H$

No $\bar{4} \in H$ but the inverse of $\bar{4}$ is $\bar{8}$ $(\bar{4}+\bar{8}=0)$ and $\bar{8} \notin H$

$\therefore$ H is **NOT** a subgroup

Good Test Quest
Hint Hint

Example: $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

Is $H = \{1, r^2, s, sr^2\}$ a subgroup of D_8?

$r^4=1,\ s^2=1,\ r^i s = sr^{-i}$

H	1	r^2	s	sr^2
1	1	r^2	s	sr^2
r^2	r^2	1	sr^2	s
s	s	sr^2	1	r^2
sr^2	sr^2	s	r^2	1

$\rightarrow r^2 s = sr^{-2} = sr^{4-2} = sr^2$

(1) closure, yes ✓

(2) identity, $1 \in H$ ✓

(3) inverses ✓

Inverses

$(r^2)^{-1} = r^2 \in H$

$(s)^{-1} = s \in H$

$(sr^2)^{-1} = sr^2 \in H$

$1^{-1} = 1 \in H$

yes $H \leq D_8$

H is called the Klein 4 group

9/14 P. 3

Def: Let G be a group with identity element e, Every group G. has <u>at least</u> two subgroups.

- The subgroup $\{e\}$ is called the trivial subgroup of G
- The subgroup G is called the improper subgroup of G, which is the entire group itself.