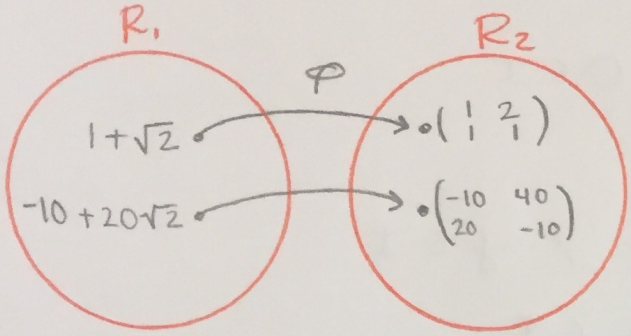
Tuesday Week 10 April 4, 2017

HW 4 (4) (b)

$R_1 = \{a+b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ $R_2 = \left\{\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{Z}\right\}$



$\varphi : R_1 \to R_2$

$\varphi(a+b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$

- Show φ is a ring isomorphism.

Proof: Let $a+b\sqrt{2}, c+d\sqrt{2} \in R_1$

Then $\varphi((a+b\sqrt{2})+(c+d\sqrt{2})) = \varphi(a+c)+b+d)\sqrt{2}) = \begin{pmatrix} a+c & 2(b+d) \\ b+d & a+c \end{pmatrix}$

and $\varphi(a+b\sqrt{2}) + \varphi(c+d\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} + \begin{pmatrix} c & 2d \\ d & c \end{pmatrix} = \begin{pmatrix} a+c & 2(b+d) \\ b+d & a+c \end{pmatrix}$

Also, $\varphi((a+b\sqrt{2})(c+d\sqrt{2})) = \varphi((ac+2bd)+(ad+bc)\sqrt{2}) = \begin{pmatrix} ac+2bd & 2(ad+bc) \\ ad+bc & ac+2bd \end{pmatrix}$

and $\varphi(a+b\sqrt{2})\varphi(c+d\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}\begin{pmatrix} c & 2d \\ d & c \end{pmatrix} = \begin{pmatrix} ac+2bd & 2(ad+bc) \\ bc+ad & 2bd+ac \end{pmatrix}$

one-to-one

Suppose $\varphi(a+b\sqrt{2}) = \varphi(c+d\sqrt{2})$

then $\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} = \begin{pmatrix} c & 2d \\ d & c \end{pmatrix}$

so $a=c$ and $b=d$

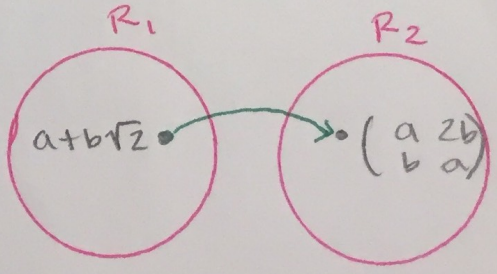
Thus, $a+b\sqrt{2} = c+d\sqrt{2}$

onto

Let $\begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$ be an arbitrary element from R_2

Then $a+b\sqrt{2} \in R_1$ and

$\varphi(a+b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$



HW 5 #8

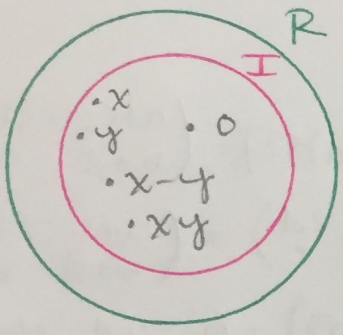
Let I be an ideal of a ring R.
Then I is a subring of R

proof:
- since I is an ideal, $0 \in I$
- let $x, y \in I$

since I is an ideal $x-y \in I$

and $xy \in I$

$x \in I$ $x \in R$



Big Theorem Thursday

Let R be a commutative ring with identity $1 \neq 0$. Let $M \neq R$ be an ideal of R. Then M is maximal iff R/M is a field.

Proof: First note that R/M is a commutative ring with identity $1+M$. and $1+M \neq 0+M$ since $M \neq R$

(we saw why this is true in the I prime $\Leftrightarrow$ R/I int. dom. proof)

($\Rightarrow$) Suppose M is maximal

Let $a+M \in R/M$ with $a+M \neq 0+M$ and $a \in R$

We need to show that $a+M$ is a unit;

i.e. has a mult. inverse.

- Let $I_a = \{m+ar \mid r \in R, m \in M\} = M + \langle a \rangle$

Claim: I_a is an ideal

Proof of claim: we know $0 \in M$ since M is an ideal

so, set $m=0$ and $r=0$ is the def of I_a to get $0 = 0 + a \cdot 0 \in I_a$

Let m_1+ar_1, $m_2+ar_2 \in I_a$ where $m_1, m_2 \in M$, $r_1, r_2 \in R$

Then $m_1 - m_2 \in M$ and $r_1 - r_2 \in R$; so $(m_1+ar_1)-(m_2+ar_2)$

$= (\underbrace{m_1-m_2}_{\text{in } M}) + a(\underbrace{r_1-r_2}_{\text{in } R}) \in I_a$

Let $m+ar \in Ia$, and $s \in R$, where $m \in M$, $r \in R$

Then $ms \in M$ since M is an ideal.

(m in M, s in R)

And $rs \in R$

so, $(m+ar)s = \underbrace{ms}_{\text{in } M} + a\underbrace{(rs)}_{\text{in } R} \in Ia$

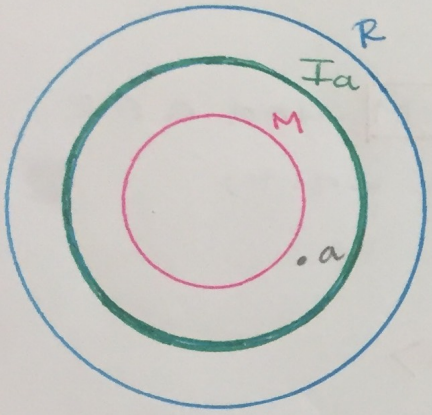
(equal)

And $s(m+ar) \in Ia$ also [claim]

claim

$M \subset Ia$

Given $m \in M$, then $m = m + a \cdot 0 \in Ia$



- since M is maximal,

  either $M = Ia$ or $R = Ia$

- Since $a+M \neq \underbrace{0+M}_{M}$ we know

  $a \notin M$ [Recall $x+I = y+I$ iff $x \in y+I$]

But $a = \underbrace{0}_{M} + a \cdot \underbrace{1}_{R} \in Ia$

Since $a \notin M$ and $a \in Ia$, we know $Ia \neq M$

Hence $Ia = R$

Since $I_a = R$ and $1 \in R$

we know $1 \in I_a$ Thus $1 = m + ar$

where $m \in M$ and $r \in R$

Thus, $1 - ar = m \in M$ so, $(1-ar) + M = \underbrace{0+M}_{M}$

so $(1+M) + (-ar+M) = 0+M$

so $1+M = ar+M$

thus, $1+M = (a+M)(r+M)$

so, $a+M$ has $r+M$ as a mult. inverse ☐

 Thursday Week 10 April 6, 2017

HW 6 #4

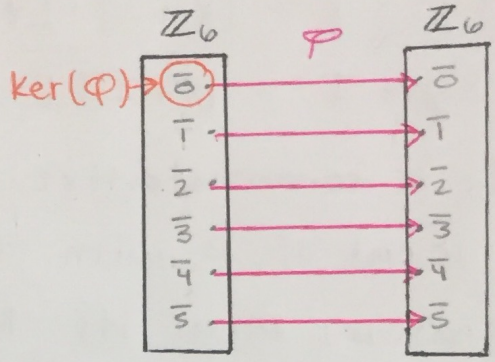
Let R be a ring. Show that $R \cong R/\{0\}$

First iso thm

$\varphi: R \to R'$ hom

$R/\ker\varphi \cong \text{im}(\varphi)$

Ex: $\varphi: \mathbb{Z}_6 \to \mathbb{Z}_6$



Proof: Let $\varphi: R \to R$ where $\varphi(x) = x$. Then φ is a homomorphism since for every $x, y \in R$ we have $\varphi(x+y) = x+y = \varphi(x) + \varphi(y)$ and $\varphi(xy) = xy = \varphi(x)\varphi(y)$

φ is 1-1 and onto and $\ker(\varphi) = \{0\}$ so $\text{im}(\varphi) = R$. By 1st iso thm. $R/\{0\} \cong R$. ☒

Def: Let R be a ring and I be an ideal of R

Let $\pi: R \to R/I$ where $\pi(r) = r + I$.

π is called the natural or canonical homomorphism.

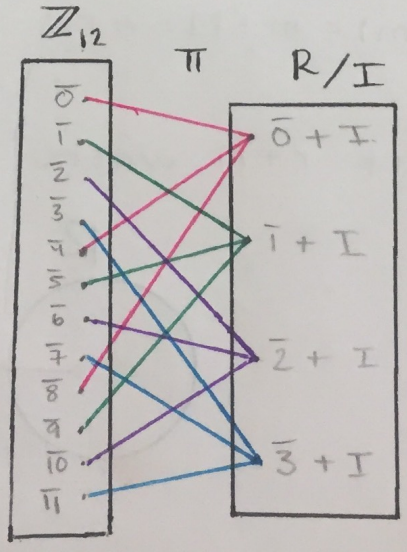
$R = \mathbb{Z}_{12}$

$I = \langle \bar{4} \rangle = \{\bar{0}, \bar{4}, \bar{8}\}$

$\bar{1} + I = \{\bar{1}, \bar{5}, \bar{9}\}$

$\bar{2} + I = \{\bar{2}, \bar{6}, \bar{10}\}$

$\bar{3} + I = \{\bar{3}, \bar{7}, \bar{11}\}$



$\pi(\bar{0}) = \bar{0} + I$

$\pi(\bar{4}) = \bar{4} + I = \bar{0} + I$

$\pi(\bar{11}) = \bar{11} + I = \bar{3} + I$

Given a ring R with ideal I

The map $\pi : R \to R/I$ given by $\pi(r) = r + I$ is a homomorphism

Proof: Let $x, y \in R$. Then

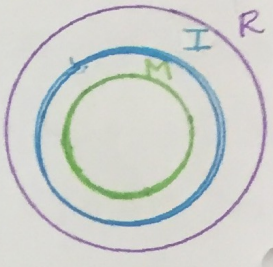
$$\pi(x+y) = (x+y) + I = [x+I] + [y+I] = \pi(x) + \pi(y)$$

and $\pi(xy) = xy + I = [x+I][y+I] = \pi(x)\pi(y)$ ▨

Theorem: Let R be a commutative ring with identity $1 \neq 0$

Let M be an ideal of R with $M \neq R$

Then M is a maximal ideal iff R/M is a field.



Proof: (➡) Last time.

(⬅) Suppose R/M is a field. Lets show that M is maximal.

Let I be an ideal of R with $M \subseteq I \subseteq R$

we must show $I = M$ or $I = R$

Let $\pi : R \to R/M$ be the canonical homomorphism given by $\pi(r) = r + M$. Apply π to $M \subseteq I \subseteq R$

to get $\pi(M) \subseteq \pi(I) \subseteq \pi(R)$

Note that $\pi(M) = \{0 + M\}$

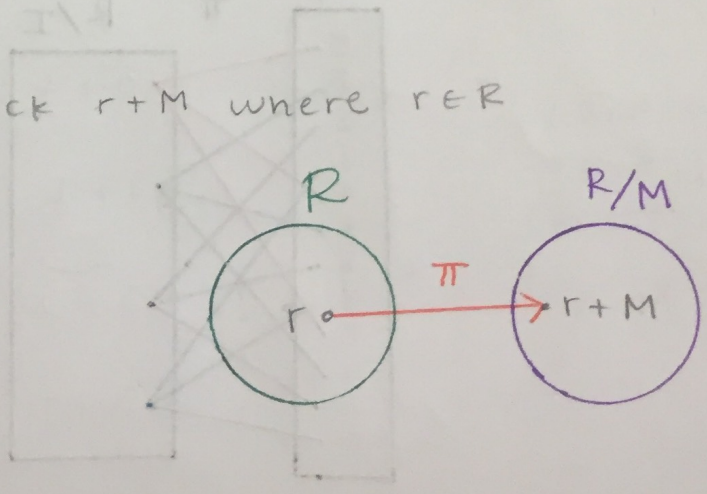
why? If $m \in M$, then $\pi(m) = m + M = 0 + M$ (since $m \in 0 + M = M$)

Also $\pi(R) = R/M$

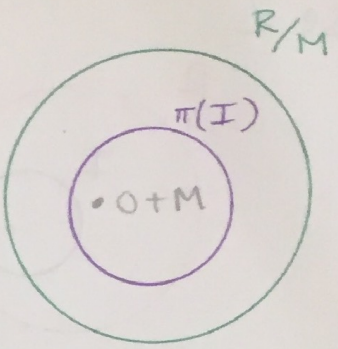
why? Because if you pick $r + M$ where $r \in R$

then $\pi(r) = r + M$

that is, π is onto



Thus, $\{0+M\} \subseteq \pi(I) \subseteq R/M$

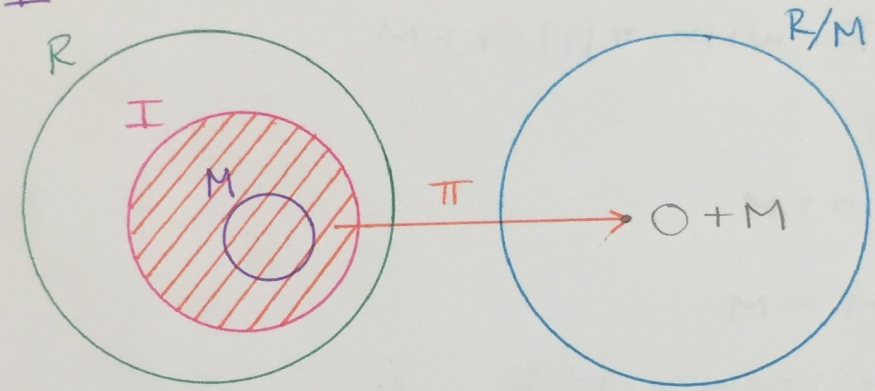


- Since R/M is a field we know it only has two ideals. They are $\{0+M\}$ and R/M

claim $\pi(I)$ is an ideal of R/M (← Prove this at home)

Therefore, $\pi(I) = \{0+M\}$ or $\pi(I) = R/M$

Case 1



Suppose $\pi(I) = \{0+M\}$

we already know $M \subseteq I$

suppose $i \in I$

Then $\pi(i) = i+M = 0+M$

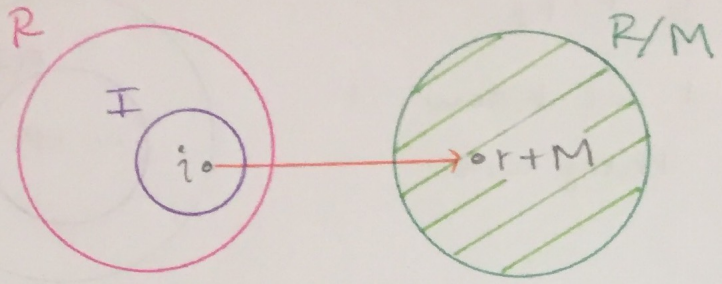
↑ def of π

so $i \in 0+M = M$

Thus $I \subseteq M$

Therefore $I = M$

Case 2



$a \equiv b \pmod{n}$

$a + n\mathbb{Z} = b + n\mathbb{Z}$

Suppose $\pi(I) = R/M$

We know $I \subseteq R$, Let's show $R \subseteq I$

Let $r \in R$ since $\pi(I) = R/M$

There exists $i \in I$ with $\pi(i) = r + M$

so $i + M = r + M$

so $(r-i) + M = 0 + M$

so $r - i \in 0 + M = M$

since $M \subseteq I$ we know $r - i \in I$

Thus, $r = \underbrace{i}_{\text{in } I} + \underbrace{(r-i)}_{\text{in } I} \in I$

since I is an ideal

so $R = I$

Thus either $I = M$ or $I = R$

so M is maximal $\blacksquare$