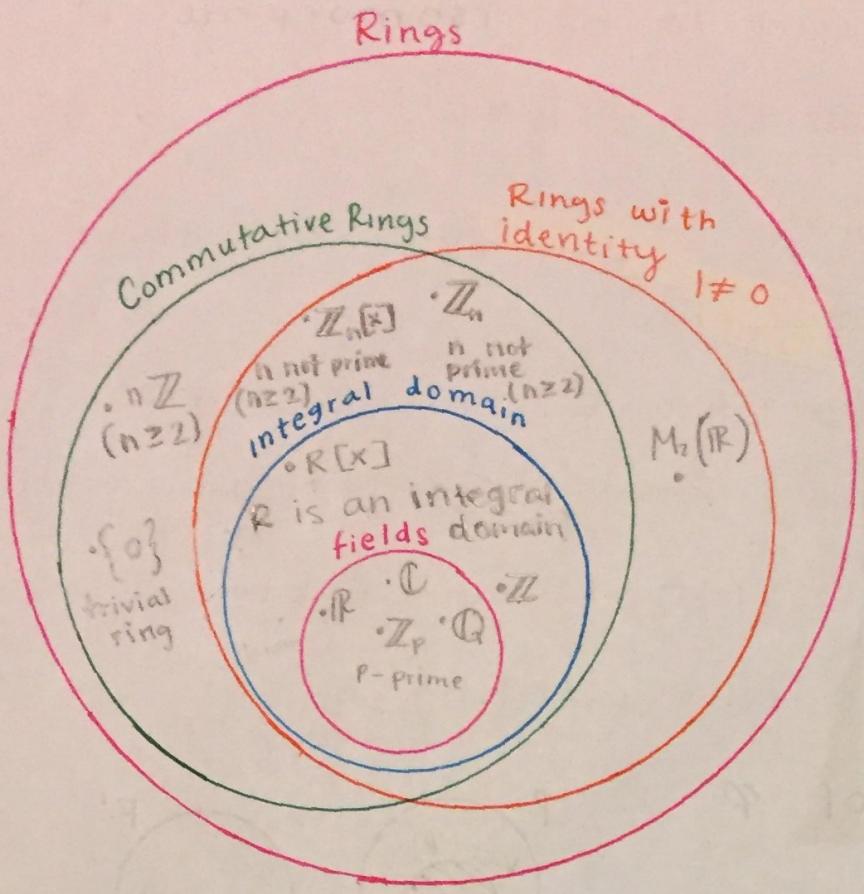
Tuesday week 4 Feb. 14, 2017

Rings



Group Homomorphism

$\langle G, * \rangle$ $\phi : G \to G'$

$\langle G', *' \rangle$ ϕ is a group homomorphism if

$\phi(g * h) = \phi(g) *' \phi(h)$

$\forall g, h \in G$

Homomorphisms of Rings

Def: Let R and R' be rings

(1) We say that $\phi : R \to R'$ is a **ring homomorphism** if $\phi(x+y) = \phi(x) + \phi(y)$

and $\phi(xy) = \phi(x)\phi(y)$ $\forall x, y \in R$

(2) If $\phi : R \to R'$ is a ring homomorphism and ϕ is 1-1 and onto then ϕ is called a **ring isomorphism**

(3) R and R' are said to be **isomorphic** if

$\exists$ a ring isomorphism $\varphi: R \to R'$

If no such isomorphism exists we say that R and R' are not isomorphic

notation: $R \cong R'$ means R and R' are isomorphic

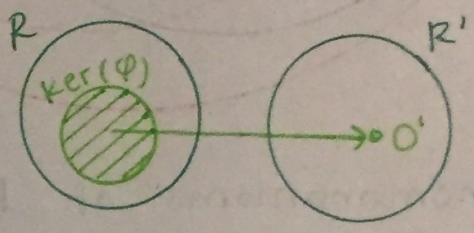
$R \ncong R'$ means R and R' are NOT isomorphic

(4) Let $\varphi: R \to R'$ be a ring homomorphism

Let $0'$ be the additive identity of R'

Then the **kernal of φ** is

$$\ker(\varphi) = \{x \in R \mid \varphi(x) = 0'\}$$



Example: $R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}$ *R is a subring of $M_2(\mathbb{R})$, see homework

Define $\varphi: R \to \mathbb{R}$ by $\varphi\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = a$

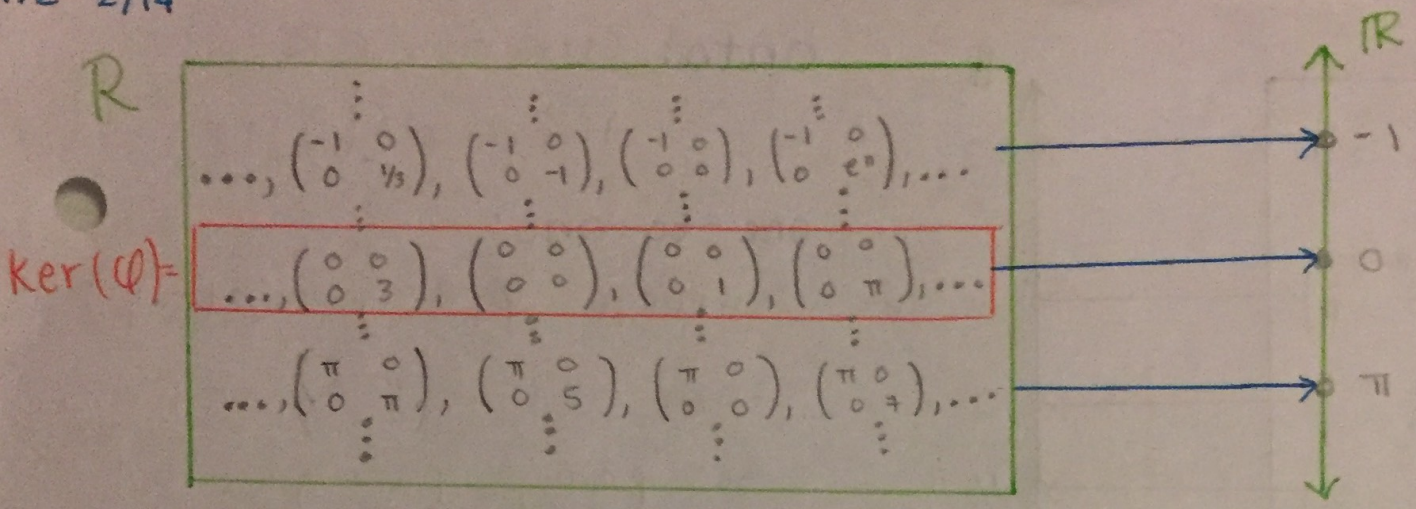
is φ a ring homomorphism?

Let $\begin{pmatrix} x & 0 \\ 0 & f \end{pmatrix}$ and $\begin{pmatrix} z & 0 \\ 0 & j \end{pmatrix}$ be in R

Then $\varphi\left(\begin{pmatrix} x & 0 \\ 0 & f \end{pmatrix} + \begin{pmatrix} z & 0 \\ 0 & j \end{pmatrix}\right) = \varphi\begin{pmatrix} x+z & 0 \\ 0 & f+j \end{pmatrix} = x + z = \varphi\begin{pmatrix} x & 0 \\ 0 & f \end{pmatrix} + \varphi\begin{pmatrix} z & 0 \\ 0 & j \end{pmatrix}$

and $\varphi\left(\begin{pmatrix} x & 0 \\ 0 & f \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & j \end{pmatrix}\right) = \varphi\begin{pmatrix} xz & 0 \\ 0 & fj \end{pmatrix} = xz = \varphi\begin{pmatrix} x & 0 \\ 0 & f \end{pmatrix}\varphi\begin{pmatrix} z & 0 \\ 0 & j \end{pmatrix}$

$\therefore$ yes it is.



φ is not 1-1 (injective)

φ is onto (surjective)

[Given $r \in \mathbb{R}$, $\varphi\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} = r$]

$$\ker(\varphi) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R \mid \varphi\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = 0 \right\}$$

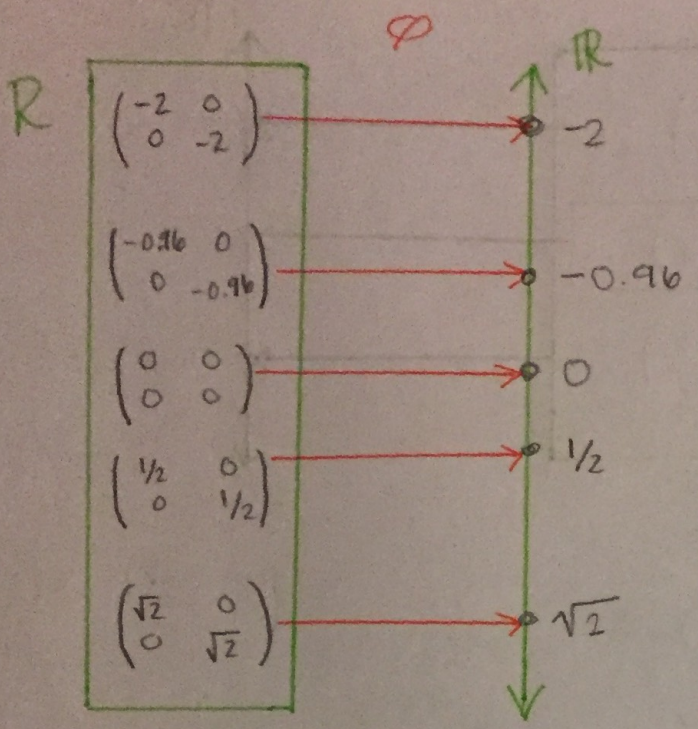
$$= \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R \mid a = 0 \right\} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \mid b \in \mathbb{R} \right\}$$

Ex: $R = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{R} \right\}$, R is a subring of $M_2(\mathbb{R})$

Define $\varphi : R \to \mathbb{R}$

$$\varphi\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a$$

-you can verify that φ is a ring homomorphism.



Onto: Given $r \in R$

then $\varphi\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} = r$

one-to-one:

suppose $\varphi\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \varphi\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$

then $a = b$

so, $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$

thus φ is an isomorphism and $R \cong \mathbb{R}$

$\ker(\varphi) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$

Lemma: Let R and R' be rings and $\varphi: R \to R'$ be a ring homomorphism,

Let 0 and $0'$ be the additive identities of R and R' then $\varphi(0) = 0'$

Proof:

Note that $\varphi(0+0) = \varphi(0)$

since φ is a ring homomorphism we have

$$\varphi(0) + \varphi(0) = \varphi(0)$$

So, $\underbrace{-\varphi(0) + \varphi(0)}_{0'} + \varphi(0) = \underbrace{-\varphi(0) + \varphi(0)}_{0'}$

Thus $\varphi(0) = 0'$ ∎

Theorem:

Let R and R' be rings and $\varphi : R \to R'$ be a ring homomorphism. φ is 1-1 iff $\ker(\varphi) = \{0\}$

Here 0 and $0'$ are the additive identities of R and R'

Proof:

($\Leftarrow$) Suppose $\ker(\varphi) = \{0\}$ we must prove that φ is 1-1.

Suppose $\varphi(x) = \varphi(y)$ for some $x, y \in R$

So $\varphi(x) - \varphi(y) = 0'$

Claim: $-\varphi(y) = \varphi(-y)$

proof: $\varphi(y) + \varphi(-y) = \varphi(y-y) = \varphi(0) = 0'$

(homo, lemma)

So $\varphi(-y) = -\varphi(y)$

Thus $\varphi(x) = \varphi(-y) = 0'$

so, $\varphi(x-y) = 0'$

Thus $x-y \in \ker(\varphi)$ since $\ker(\varphi) = \{0\}$

we have $x-y = 0$, so $x = y$

$\therefore \varphi$ is 1-1

to be continued...

Thursday Week 4 Feb. 16. 2017

Let K be a field

$K^* = K \setminus \{0\}$

$S = \{x^2 \mid x \in K^*\}$

① K^* is a group under $\cdot$

② S is a subgroup of K^*

K^* is a group
- K has 1
- K is associative under mult since K is a ring
- K^* is closed under mult b/c K is a ring and is closed under mult. and K is an integral domain so if $a, b \in K^*$ (ie $a, b \neq 0$) then $ab \in K^*$ ($ab \neq 0$)
- Let $a \in K^*$ so $a \neq 0$ since K is a field, a has an inverse under mult.

Fact

$\mathbb{Z}_p \setminus \{0\}$ is a cyclic group under mult.

Ex: $K = \mathbb{Z}_{11}$

$K^* = \mathbb{Z}_{11} \setminus \{0\}$

$K^* = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}\}$

$S = \{\bar{1}^2, \bar{2}^2, \bar{3}^2, \bar{4}^2, \bar{5}^2, \bar{6}^2, \bar{7}^2, \bar{8}^2, \bar{9}^2, \bar{10}^2\}$

$= \{\bar{1}^2, \bar{2}^2, \bar{3}^2, \bar{4}^2, \bar{5}^2, (\bar{-5})^2, (\bar{-4})^2, (\bar{-3})^2, (\bar{-2})^2, (\bar{-1})^2\}$

$= \{\bar{1}, \bar{4}, \bar{9}, \bar{5}, \bar{3}\}$ ← This is a group under mult.

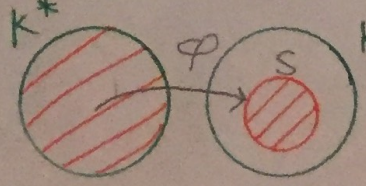
$S \leq K^*$

$S \leq K^*$

Proof: • $1 = 1^2$, so $1 \in S$

• Let $x, y \in S$, then $x = a^2$ and $y = b^2$ where $a, b \in K^*$. Then since K^* is a group we know $b^{-1} \in K^*$ and so $ab^{-1} \in K^*$ so, $xy^{-1} = a^2(b^2)^{-1} = a^2b^{-2} = \underbrace{(ab^{-1})^2}_{\text{in } K^*}$

so $xy^{-1} \in S$ ▨



$\varphi(x) = x^2$

$S = \text{im}(\varphi)$

Theorem: Let R and R' be rings.

Let $\varphi: R \to R'$ be a ring homomorphism

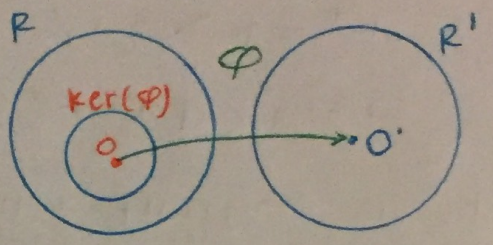
Then φ is 1-1 iff $\ker(\varphi) = \{0\}$

0 and $0'$ are the additive identities of R and R'

Proof:

($\Leftarrow$) Last Time

($\Rightarrow$) Suppose φ is 1-1



Suppose $x \in \ker(\varphi)$

Then $\varphi(x) = 0'$

Last time we had a lemma showing $\varphi(0) = 0'$

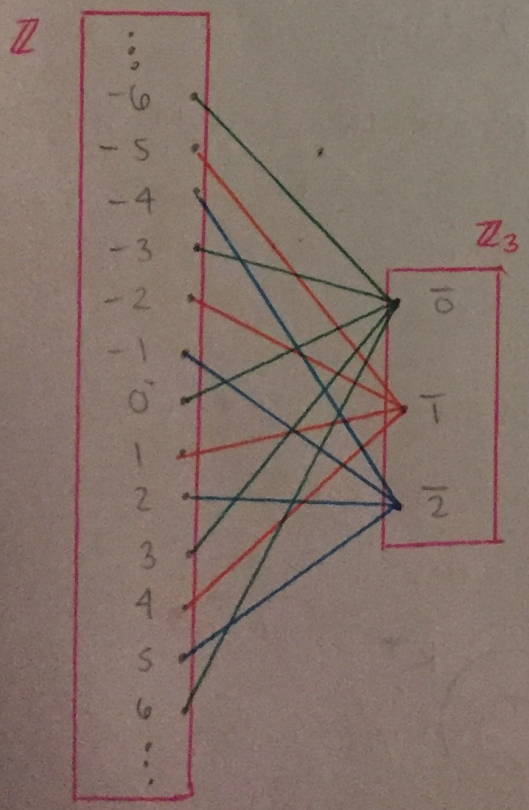
so $\varphi(x) = \varphi(0)$.

Since φ is 1-1 we must have $x = 0$.

So, $\ker(\varphi) = \{0\}$ ▨

Ex: Let $n \geq 2$, consider $\pi_n : \mathbb{Z} \to \mathbb{Z}_n$

where $\pi_n(x) = \overline{x}$



π_n is a ring homomorphism

Proof: Let $x, y \in \mathbb{Z}$

then,

$$\pi_n(x+y) = \overline{x+y} \overset{\text{def of } + \text{ in } \mathbb{Z}_n}{=} \overline{x} + \overline{y} = \pi_n(x) + \pi_n(y)$$

and

$$\pi_n(xy) = \overline{xy} = \overline{x} \cdot \overline{y} = \pi_n(x)\,\pi_n(y)$$

(def of $\cdot$ in $\mathbb{Z}_n$)

▨

$\ker(\pi_n) = n\mathbb{Z} = \{nk \mid k \in \mathbb{Z}\}$

Proof:

($\Rightarrow$) Let $x \in \ker(\pi_n)$

So $\pi_n(x) = \bar{0}$ then, $\bar{x} = \bar{0}$

hence $x \equiv 0 \pmod{n}$, so, $n \mid (x-0)$

Therefore, $x = nk$ where $k \in \mathbb{Z}$

so, $x \in n\mathbb{Z}$

Ergo $\ker(\pi_n) \subseteq n\mathbb{Z}$

($\Leftarrow$) Let $y \in n\mathbb{Z}$

Then $y = nl$ where $l \in \mathbb{Z}$

so, $\pi_n(y) = \bar{y} = \overline{nl} = \bar{n} \cdot \bar{l} = \bar{0} \cdot \bar{l} = \bar{0}$

$\bar{n} = \bar{0}$ in $\mathbb{Z}_n$

Thus, $y \in \ker(\pi_n)$

so, $n\mathbb{Z} \subseteq \ker(\pi_n)$ ▨

π_n is onto

π_n is not 1-1

HW 2/10

HW #2 Ex #3

Let R and S be integral domains

Is R x S an integral domain?

Ex: $\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(\bar{0},\bar{0}), (\bar{1},\bar{0}), (\bar{0},\bar{1}), (\bar{1},\bar{1}), (\bar{0},\bar{2}), (\bar{1},\bar{2})\}$

additive identity

mult identity

$(\bar{1},\bar{0}) \cdot (\bar{0},\bar{1}) = (\bar{1}\cdot\bar{0}, \bar{0}\cdot\bar{1}) = (\bar{0},\bar{0})$

not additive identities

$(\bar{1},\bar{0})$ and $(\bar{0},\bar{1})$ are zero divisors.

$\mathbb{Z}_2 \times \mathbb{Z}_3$ is not an integral domain.

Answ: NO, R x S is never an integral domain

Let $0_R, 0_S, 1_R, 1_S$, be the additive and mult identities of R and S. Then,

so we have .

Zero divisors

$(1_R, 0_S)$

$(0_R, 1_S)$

$\leftarrow (1_R, 0_S)\cdot(0_R, 1_S) = (0_R, 0_S)$

not $(0_R, 0_S)$

So, R x S is not an integral domain