Thursday Week 7, March 9. 2017

Quotient Rings

Let R be a ring and I be an ideal of R. Then, since R is an abelian group under $+$, we know that I is a normal subgroup of R under $+$.

4550
In an abelian group every subgroup is normal

Just like in 4550, we denote the set of the left cosets as R/I and given $x \in R$ the left cosets of x is $x+I = \{x+i \mid i \in I\}$

(4550 xI)

Example: $R = \mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$

$I = 4\mathbb{Z} = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$

Left Cosets

$0+I = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$

$1+I = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$2+I = \{\ldots, -6, -2, 2, 6, 10, \ldots\} = -2+I$

$3+I = \{\ldots, -5, -1, 3, 7, 11, \ldots\} = -125+I$

$R/I = \mathbb{Z}/4\mathbb{Z} = \{0+I, 1+I, 2+I, 3+I\}$

Recall: $a+I = b+I$
iff
$a-b \in I$
iff
$a \in b+I$

Ex: $(-125)-(3) = -128 = 4(-32) \in I$

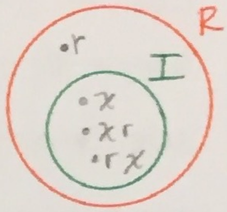
so, $\underline{-125+I} = 3+I$

$-125+I = 3 + \overbrace{\underbrace{(-128)}_{\text{in } I}}^{I} + I$

$= 3+I$

Theorem: Let R be a ring and I be an ideal of R. Then the set of left cosets

$$R/I = \{r + I \mid r \in R\}$$



is a ring with well-defined operations

$$(a+I) + (b+I) = (a+b) + I$$

$$(a+I)(b+I) = ab + I \quad \text{for any } a, b \in R$$

Note: The additive identity of R/I is $0 + I$.

The additive inverse of $a + I$ is $(-a) + I$

Example: $\mathbb{Z}/4\mathbb{Z} = \{\underbrace{0+I}_{\text{additive identity}}, \underbrace{1+I}_{\text{mult. identity}}, 2+I, 3+I\}$

$I = 4\mathbb{Z}$

$$[0+I][3+I] = (0+3) + I = 3 + I$$

$$[3+I][3+I] = [3+3] + I = 6 + I = 2 + I$$

$6 - 2 = 4 \in I = 4\mathbb{Z}$

$$[2+I][2+I] = [2 \cdot 2] + I = 4 + I = 0 + I$$

$4 - 0 = 4 \in I$

$$[3+I][-5+I] = [3 \cdot (-5)] + I = -15 + I = 1 + I$$

$-15 - 1 = -16 \in I$

or

$-15 + I = -15 + I + 16 = 1 + I$

$16 \in I$

Additive Inverse of $3 + I$

is $1 + I$ because $(1+I) + (3+I) = 4 + I = 0 + I$

Example $R = \mathbb{Z}_4 \times \mathbb{Z}_4 = \{\overset{\text{additive identity}}{(\bar{0},\bar{0})}, (\bar{0},\bar{1}), (\bar{0},\bar{2}), (\bar{0},\bar{3}), (\bar{1},\bar{0}), \overset{\text{multiplicative identity}}{(\bar{1},\bar{1})}, (\bar{1},\bar{2}),$
$(\bar{1},\bar{3}), (\bar{2},\bar{0}), (\bar{2},\bar{1}), (\bar{2},\bar{2}), (\bar{2},\bar{3}), (\bar{3},\bar{0}), (\bar{3},\bar{1}), (\bar{3},\bar{2}), (\bar{3},\bar{3})\}$

$I = \{(\bar{0},\bar{0}), (\bar{0},\bar{2}), (\bar{2},\bar{0}), (\bar{2},\bar{2})\}$ is an ideal of $\mathbb{Z}_4 \times \mathbb{Z}_4$

left coset: $\rightarrow (\bar{0},\bar{0}) + I = \{(\bar{0},\bar{0}), (\bar{0},\bar{2}), (\bar{2},\bar{0}), (\bar{2},\bar{2})\}$

$\rightarrow (\bar{0},\bar{1}) + I = \{(\bar{0},\bar{1}), (\bar{0},\bar{3}), (\bar{2},\bar{1}), (\bar{2},\bar{3})\} = (\bar{2},\bar{3}) + I$

$\rightarrow (\bar{1},\bar{0}) + I = \{(\bar{1},\bar{0}), (\bar{1},\bar{2}), (\bar{3},\bar{0}), (\bar{3},\bar{2})\}$

$\rightarrow (\bar{1},\bar{1}) + I = \{(\bar{1},\bar{1}), (\bar{1},\bar{3}), (\bar{3},\bar{1}), (\bar{3},\bar{3})\}$

$$R/I = \{\underbrace{(\bar{0},\bar{0}) + I}_{\text{additive identity}}, (\bar{1},\bar{0}) + I, (\bar{0},\bar{1}) + I, \underbrace{(\bar{1},\bar{1}) + I}_{\text{multiplicative identity}}\}$$

$$[(\bar{1},\bar{1}) + I] + [(\bar{1},\bar{1}) + I] = (\bar{2},\bar{2}) + I = (\bar{0},\bar{0}) + I$$

Example: $R = 2\mathbb{Z} = \{\dots, -8, -6, -4, -2, \overset{\text{additive identity}}{0}, 2, 4, 6, 8, \dots\}$ There is no mult. identity in $2\mathbb{Z}$

$I = 4\mathbb{Z} = \{\dots, -8, -4, 0, 4, 8, \dots\}$

left cosets: $0 + 4\mathbb{Z} = \{\dots, -8, -4, 0, 4, 8, \dots\}$

$2 + 4\mathbb{Z} = \{\dots, -6, -2, 2, 6, 10, \dots\}$

$$R/I = 2\mathbb{Z}/4\mathbb{Z} = \{\underbrace{0 + 4\mathbb{Z}}_{\text{additive identity}}, 2 + 4\mathbb{Z}\}$$

Lets make a principal ideal in $\mathbb{Z}_4 \times \mathbb{Z}_4$

$R = \mathbb{Z}_4 \times \mathbb{Z}_4 = \{(\bar{0},\bar{0}), (\bar{0},\bar{1}), (\bar{0},\bar{2}), (\bar{0},\bar{3}), (\bar{1},\bar{0}), (\bar{1},\bar{1}), (\bar{1},\bar{2}), (\bar{1},\bar{3}),$
$(\bar{2},\bar{0}), (\bar{2},\bar{1}), (\bar{2},\bar{2}), (\bar{2},\bar{3}), (\bar{3},\bar{0}), (\bar{3},\bar{1}), (\bar{3},\bar{2}), (\bar{3},\bar{3})\}$

Let's choose $(\bar{2},\bar{3})$

$\underbrace{\langle(\bar{2},\bar{3})\rangle}_{\text{Ideal generated by } (\bar{2},\bar{3})} = R(\bar{2},\bar{3}) = \{(\bar{0},\bar{0})(\bar{2},\bar{3}), (\bar{0},\bar{1})(\bar{2},\bar{3}), (\bar{0},\bar{2})(\bar{2},\bar{3}), \ldots, (\bar{3},\bar{3})(\bar{2},\bar{3})\}$

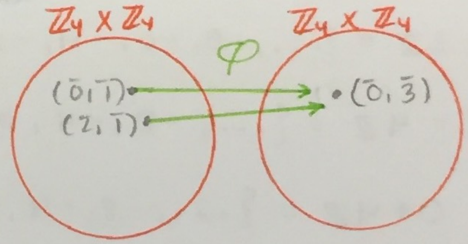
$= \{(\bar{0},\bar{0}), (\bar{0},\bar{3}), (\bar{0},\bar{2}), (\bar{0},\bar{1}), (\bar{2},\bar{0}), (\bar{2},\bar{3}), (\bar{2},\bar{2}), (\bar{2},\bar{1}),$
$(\bar{0},\bar{0}), (\bar{0},\bar{3}), (\bar{0},\bar{2}), (\bar{0},\bar{1}), (\bar{2},\bar{0}), (\bar{2},\bar{3}), (\bar{2},\bar{2}), (\bar{2},\bar{1})\}$

Principal ideal of $\mathbb{Z}_4 \times \mathbb{Z}_4$ $= \{(\bar{0},\bar{0}), (\bar{0},\bar{3}), (\bar{0},\bar{2}), (\bar{0},\bar{1}), (\bar{2},\bar{0}), (\bar{2},\bar{3}), (\bar{2},\bar{2}), (\bar{2},\bar{1})\}$

$\varphi : \mathbb{Z}_4 \times \mathbb{Z}_4 \to \mathbb{Z}_4 \times \mathbb{Z}_4$

$\varphi(\bar{a},\bar{b}) = (\bar{a},\bar{b})(\bar{2},\bar{3})$

φ is a group homomorphism under +



$R = 2\mathbb{Z}$

$\langle 4\rangle = (2\mathbb{Z})\cdot 4$

$\langle 4\rangle = \{\ldots, -24, -16, -8, 0, 8, 16, 24, \ldots\}$

principal ideal of $2\mathbb{Z}$