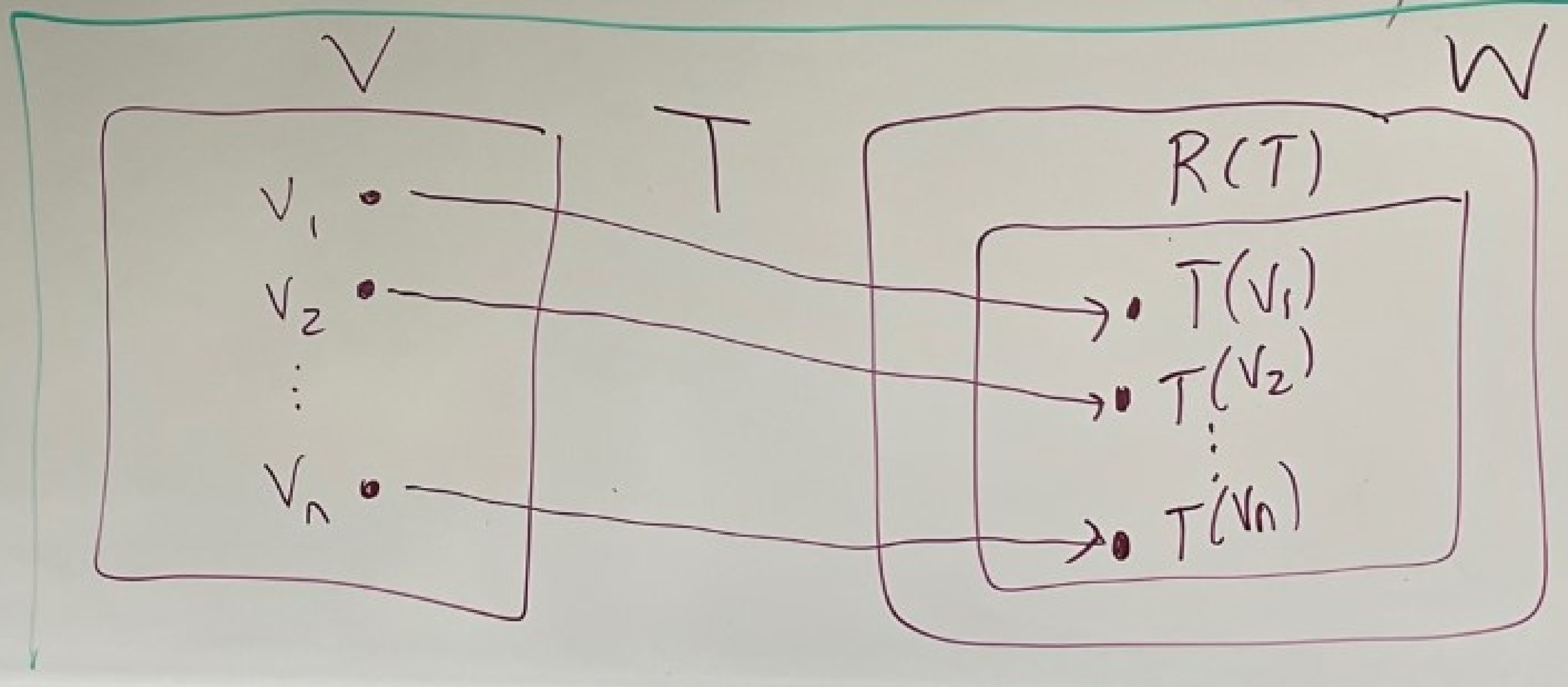
Lemma: Let V and W be vector spaces over a field F.

Let $T : V \to W$ be a linear transformation.

If $v_1, v_2, \ldots, v_n \in V$ and $\text{span}(\{v_1, v_2, \ldots, v_n\}) = V$

then $\underbrace{R(T)}_{\text{range of } T} = \text{span}(\{T(v_1), T(v_2), \ldots, T(v_n)\})$



Proof:

Let $y \in R(T)$.

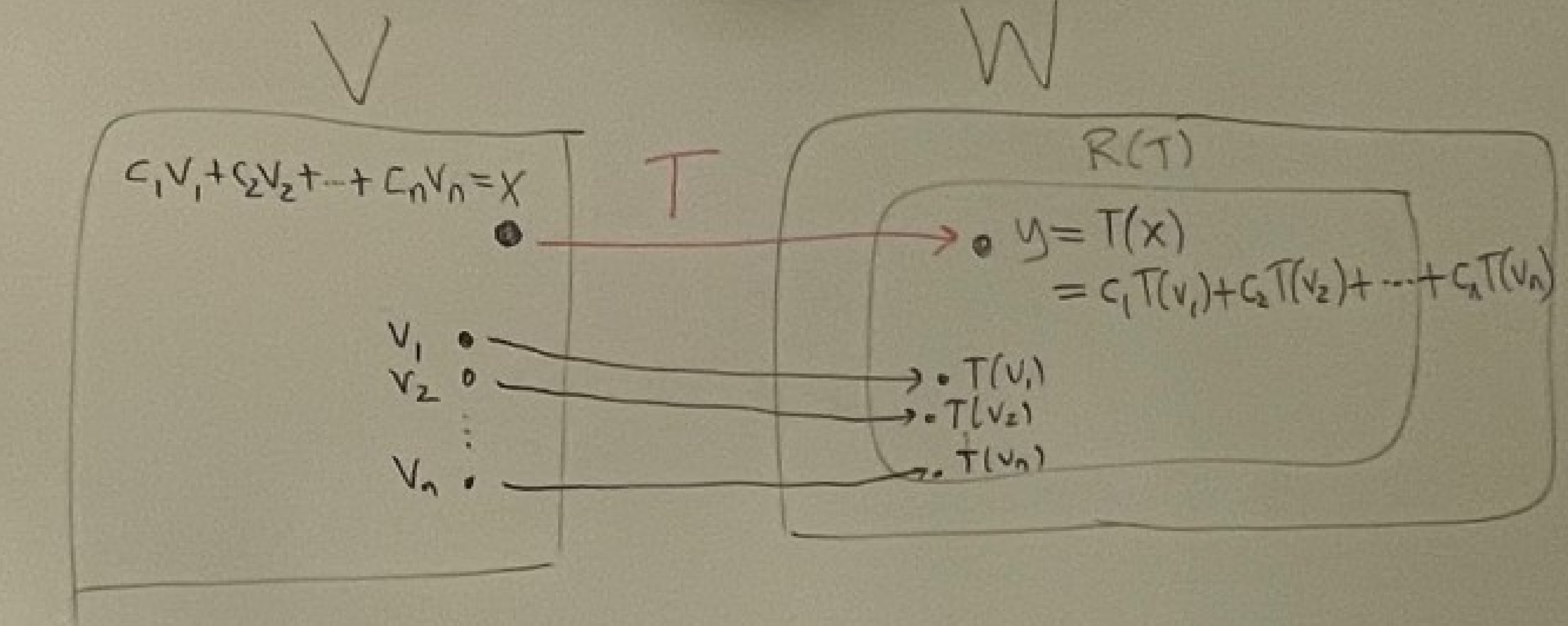
We need to write y as a linear combination of $T(v_1), T(v_2), \ldots, T(v_n)$.

Since $y \in R(T)$ there exists $x \in V$ where $T(x) = y$.

Since $x \in V$ and V is spanned by $v_1, v_2, \ldots, v_n$ we can write

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

where $c_1, c_2, \ldots, c_n \in F$.

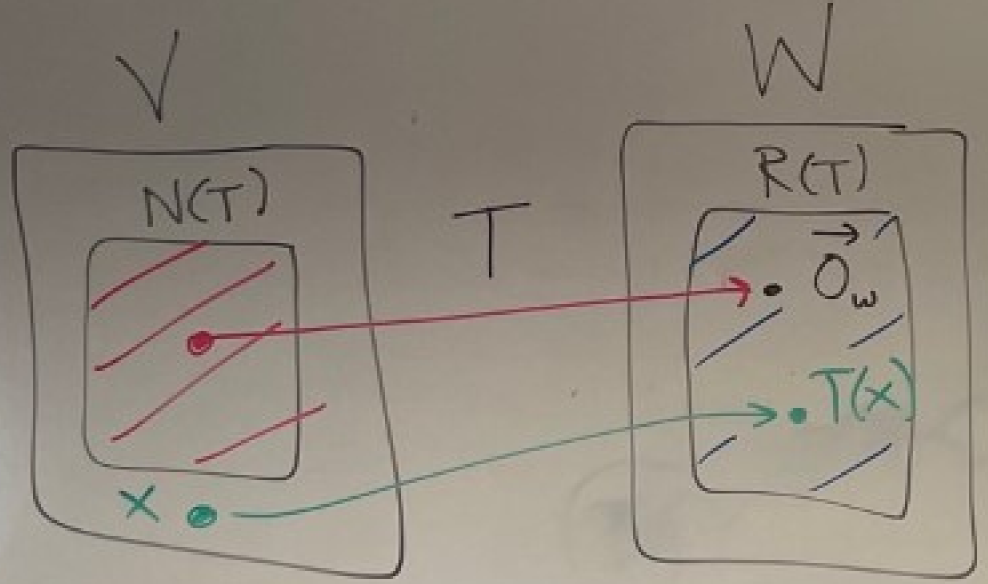


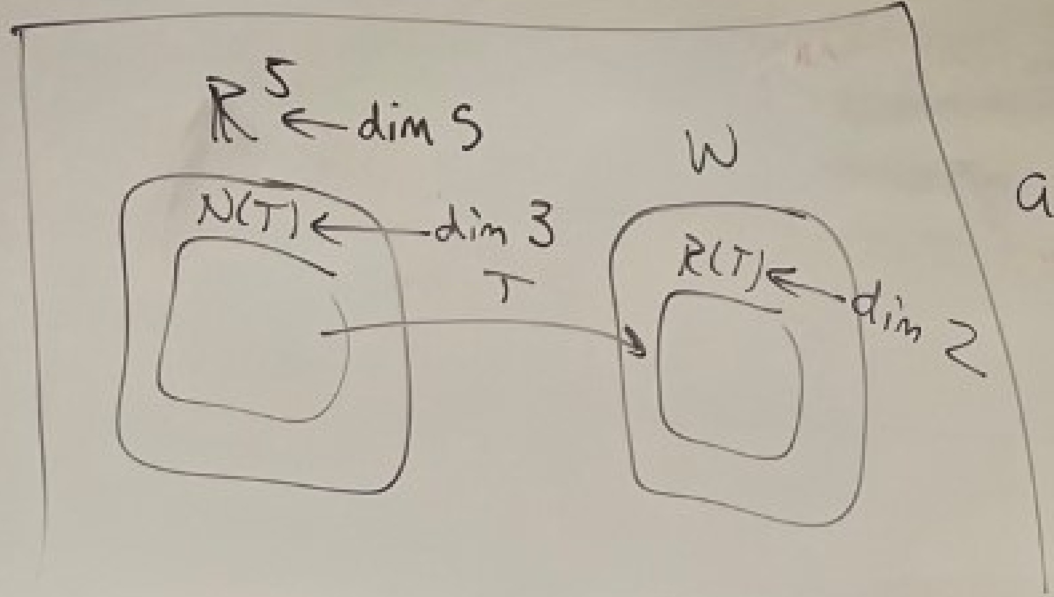
Thus,

$$y = T(x) = T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$$

$$\underset{\text{T is linear}}{=} c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$$

So, $y \in \text{span}(\{T(v_1), T(v_2), \ldots, T(v_n)\})$. $\blacksquare$





Theorem: (Rank-Nullity Theorem)

Let V and W be vector spaces over a field F.
Let $T: V \to W$ be a linear transformation.
If V is finite dimensional, then

① $N(T)$ is finite-dimensional

② $R(T)$ is finite-dimensional

and ③ $\dim(V) = \underbrace{\dim(N(T))}_{\text{nullity}(T)} + \underbrace{\dim(R(T))}_{\text{rank}(T)}$.

Proof: Let $n = \dim(V)$.

We proved previously that $N(T)$ is a subspace of V.

By a previous theorem this implies $N(T)$ is finite-dimensional and if $k = \dim(N(T))$ then $k \le n$.

Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for $N(T)$.

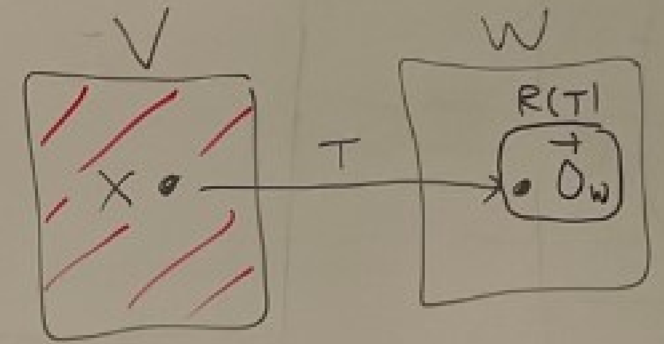
Let $\vec{0}_V$ and $\vec{0}_W$ be the zero vectors of V and W.

Recall: $T(\vec{0}_V) = \vec{0}_W$.

We break the proof into 2 cases.

Case 1: Suppose $R(T) = \{\vec{0}_W\}$.

This means $T(x) = \vec{0}_W$ for all $x \in V$.



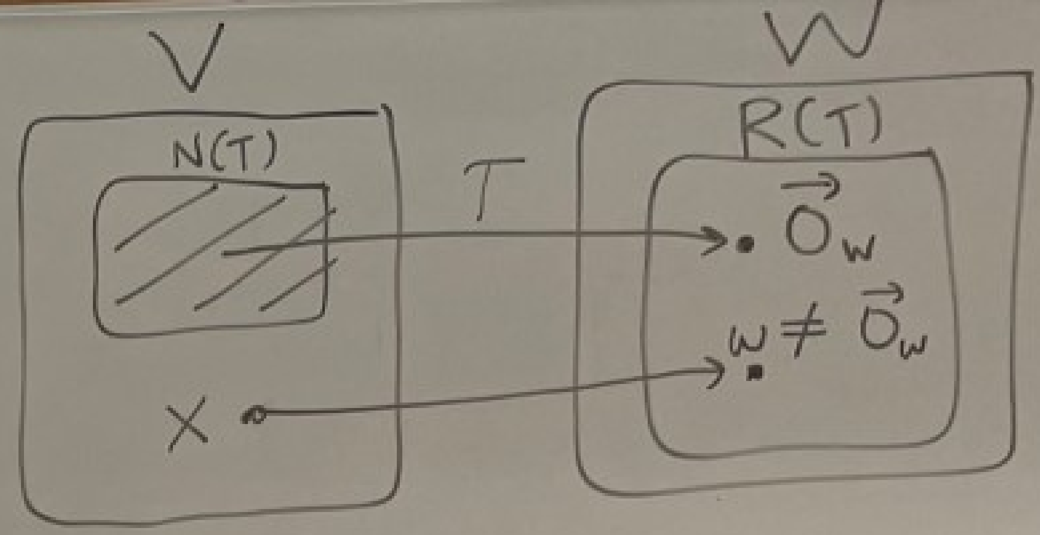
Then, $R(T) = \{\vec{0}_W\}$ has dimension 0. So, $\dim(R(T)) = 0$.

And $N(T) = V$ so $\dim(V) = \dim(N(T))$.

Thus, $\dim(V) = \underbrace{\dim(N(T))}_{\dim(V)} + \underbrace{\dim(R(T))}_{0}$.

Case 2: Suppose $R(T) \neq \{\vec{0}_W\}$

This means there is some vector $w \neq \vec{0}_W$ where $w \in R(T)$.

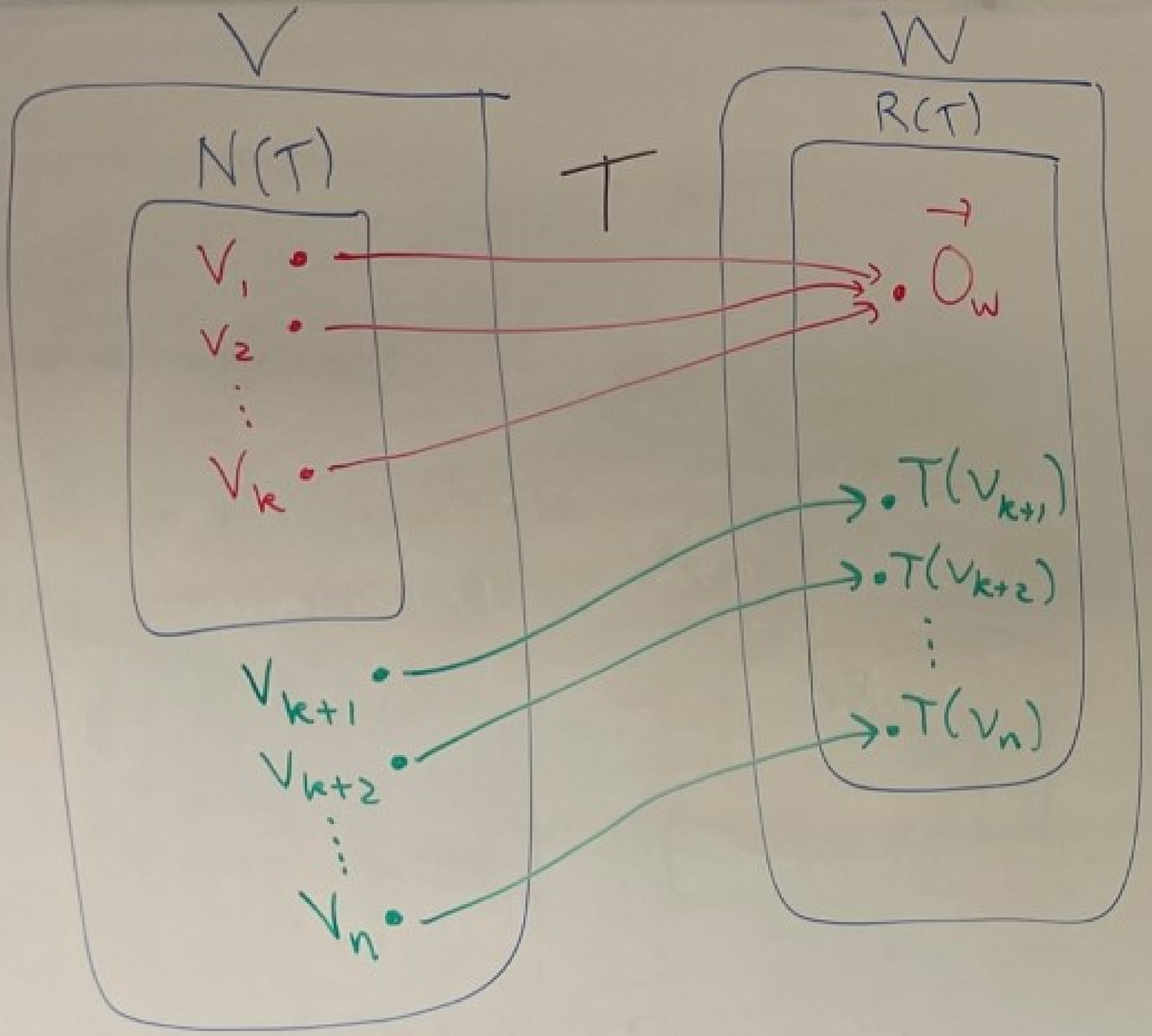


So, $N(T) \neq V$

By HW 2 #9 we can extend the basis for $N(T)$ into a basis for V.

That is, there exist vectors $v_{k+1}, v_{k+2}, \dots, v_n \in V - N(T)$

$$\{\underbrace{v_1, v_2, \dots, v_k}_{\text{basis for } N(T)}, \underbrace{v_{k+1}, v_{k+2}, \dots, v_n}_{\text{not in } N(T)}\} \text{ is a basis for } V.$$



Let $\beta' = \{T(v_{k+1}), T(v_{k+2}), \dots, T(v_n)\}$.

We will show that β' is a basis for $R(T)$.

If we do this we will be done because that will imply that

$$\begin{aligned}
\dim(V) &= n \\
&= k + (n-k) \\
&= \dim(N(T)) + \left(\begin{matrix}\#\text{ elements} \\ \text{in } \beta'\end{matrix}\right) \\
&= \dim(N(T)) + \dim(R(T)).
\end{aligned}$$

By the lemma since $\beta = \{v_1, v_2, \dots, v_k, v_{k+1}, v_{k+2}, \dots, v_n\}$ spans V we have

$$R(T) = \operatorname{span}(\{\underbrace{T(v_1)}_{\vec{0}_W}, \underbrace{T(v_2)}_{\vec{0}_W}, \dots, \underbrace{T(v_k)}_{\vec{0}_W}, T(v_{k+1}), T(v_{k+2}), \dots, T(v_n)\})$$

$$= \operatorname{span}(\{\vec{0}_W, \vec{0}_W, \dots, \vec{0}_W, T(v_{k+1}), T(v_{k+2}), \dots, T(v_n)\})$$

$$= \operatorname{span}(\{T(v_{k+1}), T(v_{k+2}), \dots, T(v_n)\})$$

$$= \operatorname{span}(\beta')$$

So, β' spans $R(T)$.

Now we want to show that β' is a linearly independent set.

Suppose

$$c_{k+1} T(v_{k+1}) + c_{k+2} T(v_{k+2}) + \cdots + c_n T(v_n) = \vec{0}_W$$

Since T is linear we have that

$$T(c_{k+1} v_{k+1} + c_{k+2} v_{k+2} + \cdots + c_n v_n) = \vec{0}_W$$

So, $c_{k+1} v_{k+1} + c_{k+2} v_{k+2} + \cdots + c_n v_n \in N(T)$.

Since $v_1, v_2, \ldots, v_k$ are a basis for $N(T)$

$\Rightarrow$ we can write

$$c_{k+1}v_{k+1} + c_{k+2}v_{k+2} + \cdots + c_n v_n = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

for some $c_1, c_2, \ldots, c_k \in F$.

Thus,

$$(-c_1)v_1 + (-c_2)v_2 + \cdots + (-c_k)v_k + c_{k+1}v_{k+1} + c_{k+2}v_{k+2} + \cdots + c_n v_n = \vec{0}_V$$

But $v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n$ is a basis for V

So $v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n$ are lin. ind.

Thus in the above eqn $(-c_1) = (-c_2) = \cdots (-c_k) = c_{k+1} = c_{k+2} = \cdots = c_n = 0$

Thus, $c_{k+1} = c_{k+2} = \cdots = c_n = 0$.

So, β' is a lin. ind. set and thus is a basis for $R(T)$. $\boxtimes$