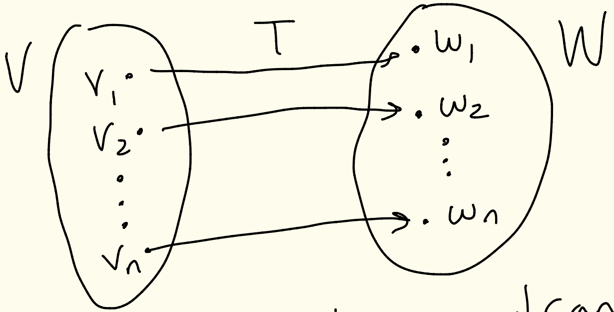
Theorem: Let V and W be vector spaces over a field F. Suppose that V is finite-dimensional and $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V.

$\boxed{\text{part 1}}$ Let $w_1, w_2, \ldots, w_n \in W$.

① There exists a unique linear transformation $T : V \to W$ where $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$



this unique linear transformation is given by the formula

$$T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1w_1 + c_2w_2 + \cdots + c_nw_n \qquad (*)$$

② T given above is an isomorphism iff $\beta' = \{w_1, w_2, \ldots, w_n\}$ is a basis for W.

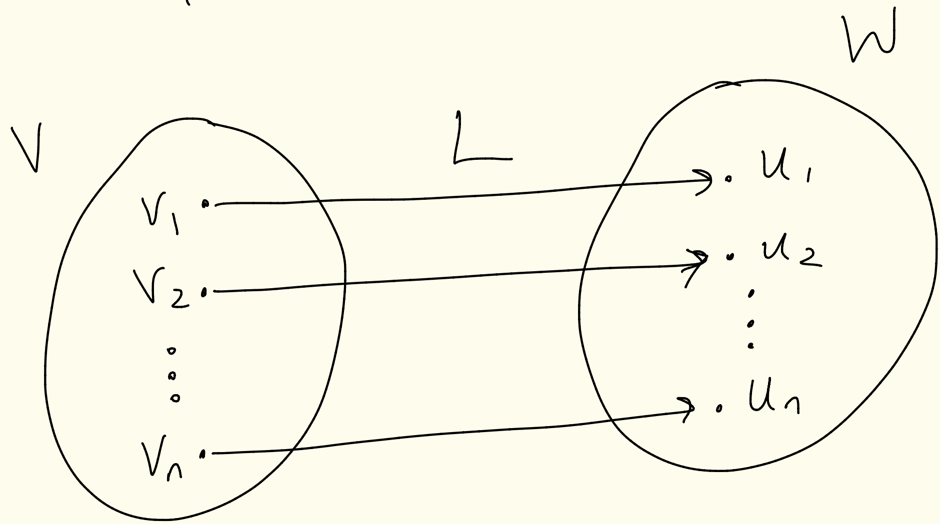
part 2

All linear transformations between V and W are constructed as in ① above. That is, if $L: V \to W$ is a linear transformation, set

$$u_i = L(v_i) \quad \text{for } i = 1, 2, \ldots, n$$

and then the formula for L is

$$L(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$$
$$= c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$



proof: $\boxed{\text{part 1}}$

① Let T be defined by $(*)$.

That is,

$$T(c_1 v_1 + \dots + c_n v_n) = c_1 w_1 + \dots + c_n w_n$$

for any $c_i \in F$.

Let's show T is a linear transformation and $T(v_i) = w_i$ for all i.

Why is T linear?

Let $x, y \in V$ and $\alpha, \delta \in F$.

Since β is a basis for V, we can write $x = e_1 v_1 + \dots + e_n v_n$ and $y = d_1 v_1 + \dots + d_n v_n$ where $e_i, d_i \in F$. Then,

$$T(\alpha x + \delta y)$$

$$= T\big(\alpha(e_1 v_1 + \dots + e_n v_n) + \delta(d_1 v_1 + \dots + d_n v_n)\big) =$$

$$= T\big((\alpha e_1 + \delta d_1) v_1 + \dots + (\alpha e_n + \delta d_n) v_n\big) =$$

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$$= T\left((\alpha e_1+\delta d_1)v_1+\cdots+(\alpha e_n+\delta d_n)v_n\right)$$

$$\overset{(*)}{=} (\alpha e_1+\delta d_1)w_1+\cdots+(\alpha e_n+\delta d_n)w_n$$

$$= \alpha e_1 w_1+\cdots+\alpha e_n w_n + \delta d_1 w_1+\cdots+\delta d_n w_n$$

$$= \alpha(e_1 w_1+\cdots+e_n w_n) + \delta(d_1 w_1+\cdots+d_n w_n)$$

$$\overset{(*)}{=} \alpha T(e_1 v_1+\cdots+e_n v_n) + \delta T(d_1 v_1+\cdots+d_n v_n)$$

$$= \alpha T(x)+\delta T(y).$$

So, T is linear.

Also,

$$T(v_1)=T(1\cdot v_1+0\cdot v_2+\cdots+0\cdot v_n)=1\cdot w_1=w_1$$

$$\vdots$$

$$T(v_n)=T(0\cdot v_1+0\cdot v_2+\cdots+1\cdot v_n)=1\cdot w_n=w_n$$

So, $T(v_i)=w_i$ for all i.

Why is T unique?

Suppose $S: V \to W$ is another linear transformation with $S(v_i) = w_i$ for $i = 1, 2, \ldots, n$.

Let $x \in V$.

Then, since β is a basis for V,

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

And,

$$
\begin{aligned}
S(x) &= S(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) \\
&\overset{\text{S is linear}}{=} c_1 S(v_1) + c_2 S(v_2) + \cdots + c_n S(v_n) \\
&\overset{S(v_i) = w_i}{=} c_1 w_1 + c_2 v_2 + \cdots + c_n w_n \\
&= T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) \\
&\overset{\text{def of } T}{=} T(x)
\end{aligned}
$$

So, $S = T$ on V.

So, T is the unique linear transf. with $T(v_i) = w_i \; \forall i$

② T defined by (*) is an isomorphism iff $\beta' = \{w_1, w_2, \ldots, w_n\}$ is a basis for W.

($\Leftarrow$) Suppose β' is a basis for W. Let's show that T defined by (*) is 1-1 and onto, and hence an isomorphism

$\boxed{1\text{-}1}$: Suppose $T(x) = T(y)$ for some $x, y \in V$.

Since β is a basis for V, $x = c_1 v_1 + \cdots + c_n v_n$ and $y = d_1 v_1 + \cdots + d_n v_n$ for $c_i, d_i \in F$.

Since $T(x) = T(y)$, by def of T, we have

$$\underbrace{c_1 w_1 + \cdots + c_n w_n}_{T(x)} = \underbrace{d_1 w_1 + \cdots + d_n w_n}_{T(y)}$$

So, $(c_1 - d_1) w_1 + \cdots + (c_n - d_n) w_n = \vec{0}$

By assumption, β' is a lin. ind. set, so $0 = c_1 - d_1 = c_2 - d_2 = \cdots = c_n - d_n$

So, $c_1 = d_1, c_2 = d_2, \dots, c_n = d_n$
and hence

$$x = c_1 v_1 + \dots + c_n v_n = d_1 v_1 + \dots + d_n v_n = y.$$

$\boxed{\text{onto}}$: We need to show $R(T) = W$.
By a previous thm, since $\beta = \{v_1, v_2, \dots, v_n\}$
spans V, we know $R(T) = \text{span}(\{T(v_1), \dots, T(v_n)\})$.
So,

$$R(T) = \text{span}(\{T(v_1), \dots, T(v_n)\})$$
$$= \text{span}(\{w_1, \dots, w_n\})$$
$$= W.$$

we are assuming $\beta' = \{w_1, \dots, w_n\}$ is a basis for W

So, T is onto W.

Thus, T is an isomorphism.

($\Rightarrow$) Now suppose T is an isomorphism, ie 1-1 and onto.
Let's show β' is a basis for W.

Since T is onto, $R(T) = W$.

Therefore,

$$W = R(T) = \text{span}(\{T(v_1), \dots, T(v_n)\})$$
$$= \text{span}(\{w_1, \dots, w_n\})$$

So, β' spans W.

Is β' a lin. ind. set?

Suppose

$$d_1 w_1 + \dots + d_n w_n = \vec{0}_W$$

where $d_i \in F$.

Since T is 1-1 and onto, T^{-1} exists and is linear (from Monday) and $T^{-1}(w_i) = v_i$ for $i = 1, \dots, n$.

Since T^{-1} is linear, $T^{-1}(\vec{0}_w)=\vec{0}_v$.
So,

$$\vec{0}_v = T^{-1}(\vec{0}_w) = T^{-1}(d_1w_1 + \dots + d_nw_n)$$
$$= d_1T^{-1}(w_1) + \dots + d_nT^{-1}(w_n)$$
$$= d_1v_1 + \dots + d_nv_n$$

Since $\beta = \{v_1, \dots, v_n\}$ is a basis and $\vec{0}_v = d_1v_1 + \dots + d_nv_n$
we get $d_1 = d_2 = \dots = d_n = 0$.

Thus, β' is a lin. ind. set.
since if $d_1w_1 + \dots + d_nw_n = \vec{0}_w$
then $d_1 = d_2 = \dots = d_n = 0$.

So, β' is a basis for W.

part 2

Suppose L is a linear transformation and $u_i = L(v_i)$ for $i = 1, 2, \ldots, n$.

Then,

$$L(c_1 v_1 + \cdots + c_n v_n)$$

$$= c_1 L(v_1) + \cdots + c_n L(v_n)$$

$$= c_1 u_1 + \cdots + c_n u_n.$$

L is linear

$\blacksquare$