Proof from last time continued...

Let V and W be finite dimensional vector spaces over a field F.

Suppose $\dim(V) = \dim(W)$.

We need to prove $V \cong W$.

Let $n = \dim(V) = \dim(W)$.

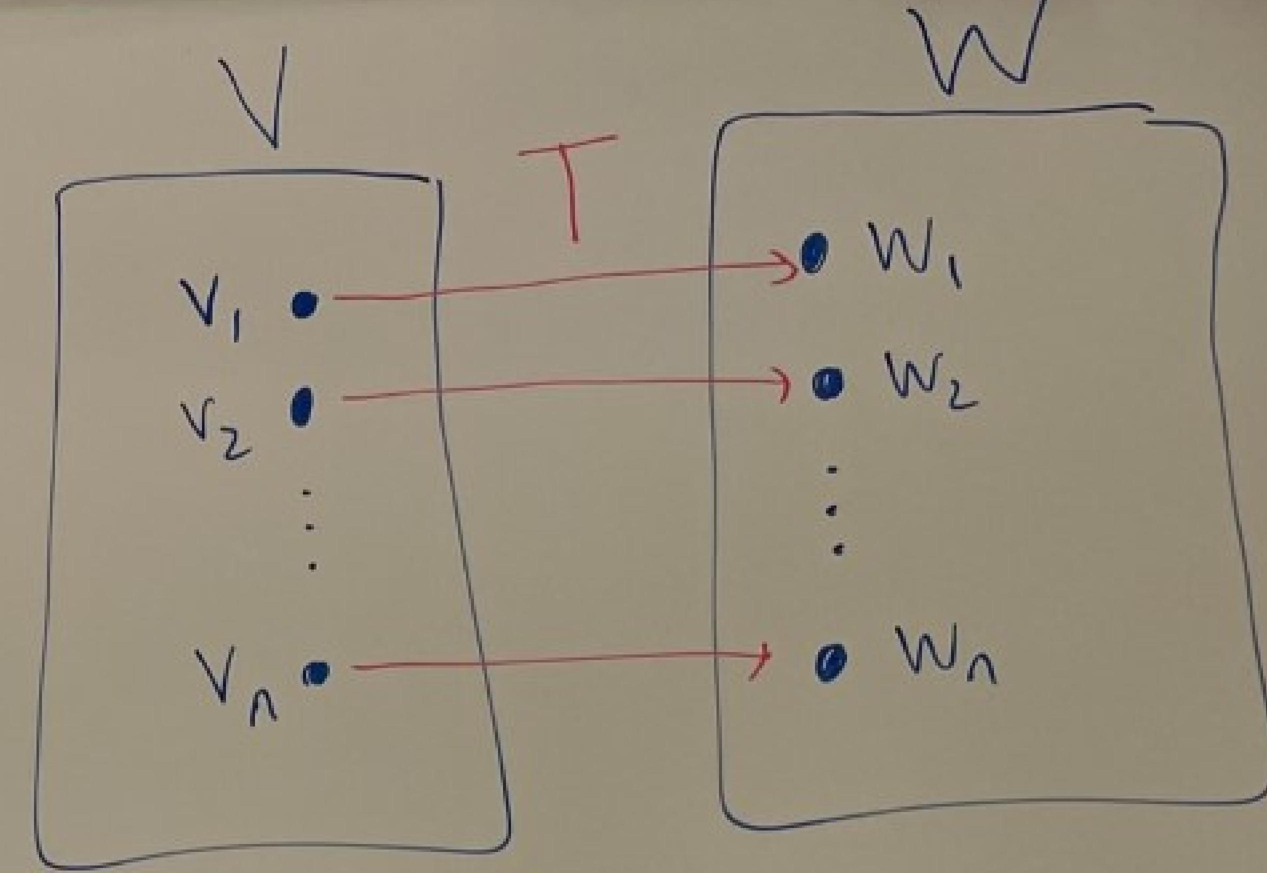
This implies there exists a basis

$$\beta = \{v_1, v_2, \ldots, v_n\} \text{ for } V$$

and a basis

$$\gamma = \{w_1, w_2, \ldots, w_n\} \text{ for } W.$$

Let's make an isomorphism $T: V \to W$ using the theorem from Monday.



Set

$T(v_1) = w_1, \; T(v_2) = w_2, \dots, T(v_n) = w_n$

and

$$T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) = \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n .$$

By thm from Monday this is a linear transformation.

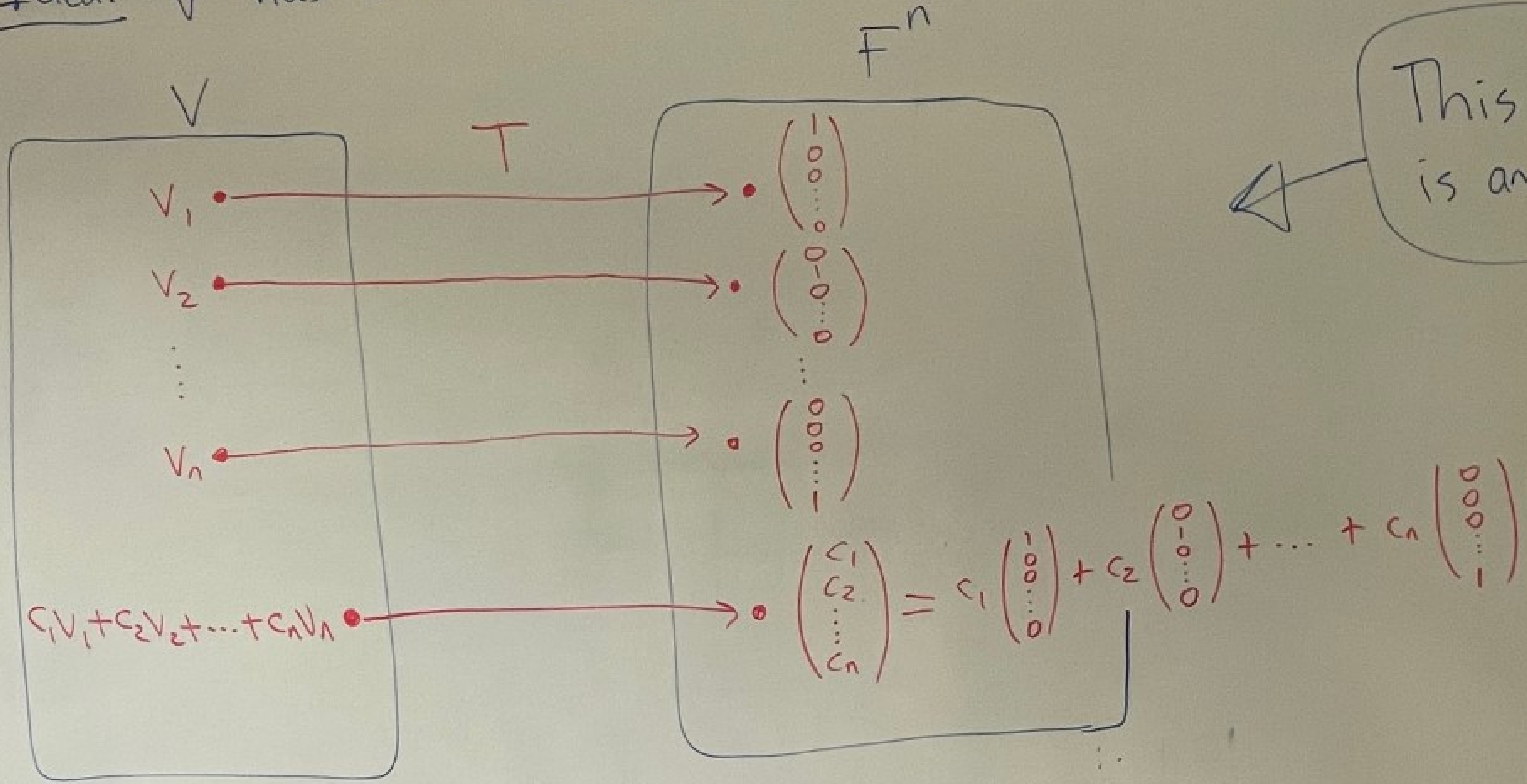
Since $w_1, w_2, \dots, w_n$ are a basis for W by Monday's theorem T is an isomorphism.

So, $V \cong W$. $\square$

Corollary: Let V be a finite-dimensional vector space over a field F.
If $\dim(V)=n$, then $V \cong F^n$.

proof: Since $\dim(V)=n=\dim(F^n)$
by the previous theorem, $V \cong F^n$ $\blacksquare$

Idea: V has basis $\{v_1, v_2, \dots, v_n\}$



This T is an isomorphism

Topic 4 – The matrix of a linear transformation

Def: Let V be a vector space over a field F.
Suppose $\{v_1, v_2, \ldots, v_n\}$ is a basis for V.
We write $\beta = [v_1, v_2, \ldots, v_n]$ to mean that
β is an ordered basis for V, that
is the order of the vectors in β is given and fixed.

Def: Let V be a vector space over a field F.
Let $\beta = [v_1, v_2, \dots, v_n]$ be an ordered basis for V.
Then given $v \in V$ we can write v uniquely in the form

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

We write

$$[v]_\beta = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

We call $[v]_\beta$ the coordinates of v with respect to β.

Note $[v]_\beta \in F^n$.

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$

You can check that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are linearly independent.

Since we have 2 lin. ind. vectors in a 2-dimensional space $\mathbb{R}^2$, they must be a basis for $\mathbb{R}^2$.

So, $\beta = \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$ is an ordered basis for $\mathbb{R}^2$.

Pick $v = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ in $\mathbb{R}^2$.

I randomly picked this v as an example

Let's find $[v]_\beta$.

We need to solve:

$$\underbrace{\begin{pmatrix}5\\4\end{pmatrix}}_{v} = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}-1\\1\end{pmatrix}$$

This gives: $\begin{pmatrix}5\\4\end{pmatrix} = \begin{pmatrix}c_1-c_2\\2c_1+c_2\end{pmatrix}$

This gives:

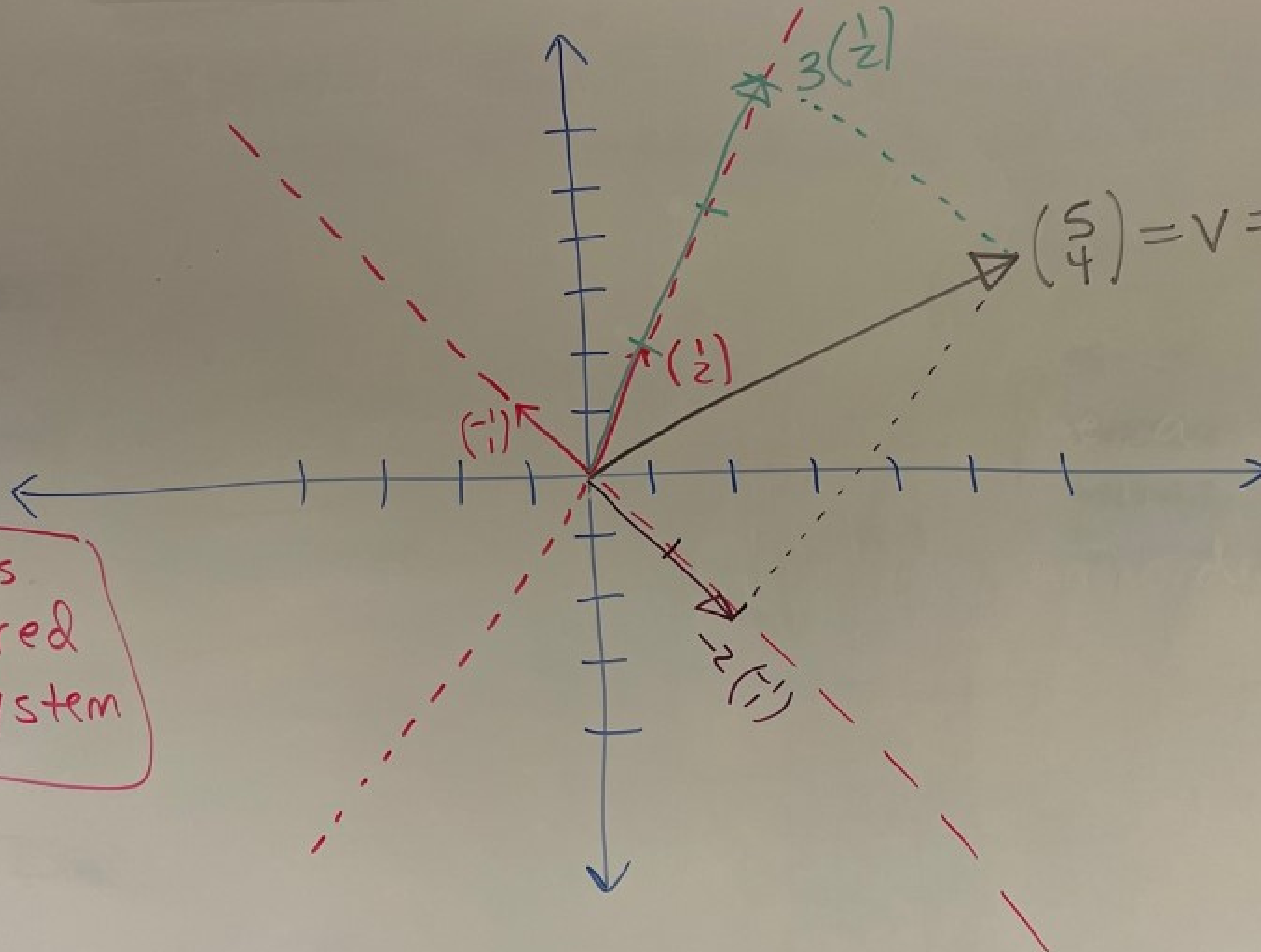
$$\boxed{\begin{aligned}c_1-c_2&=5\\2c_1+c_2&=4\end{aligned}}$$

$$\left(\begin{array}{cc|c}1&-1&5\\2&1&4\end{array}\right) \xrightarrow{-2R_1+R_2\to R_2} \left(\begin{array}{cc|c}1&-1&5\\0&3&-6\end{array}\right)$$

$$\xrightarrow{\frac{1}{3}R_2\to R_2} \left(\begin{array}{cc|c}1&-1&5\\0&1&-2\end{array}\right)$$

$$\boxed{\begin{aligned}c_1-c_2&=5\\c_2&=-2\end{aligned}} \longrightarrow \boxed{\begin{aligned}&c_2=-2\\&c_1=5+c_2=5-2=3\end{aligned}} \to \text{So, } \boxed{\begin{pmatrix}5\\4\end{pmatrix} = 3\begin{pmatrix}1\\2\end{pmatrix} - 2\begin{pmatrix}-1\\1\end{pmatrix}}$$

Thus, $[v]_\beta = \left[\begin{pmatrix}5\\4\end{pmatrix}\right]_\beta = \begin{pmatrix}3\\-2\end{pmatrix}$



$$\begin{pmatrix}5\\4\end{pmatrix} = v = 3\begin{pmatrix}1\\2\end{pmatrix} - 2\begin{pmatrix}-1\\1\end{pmatrix}$$

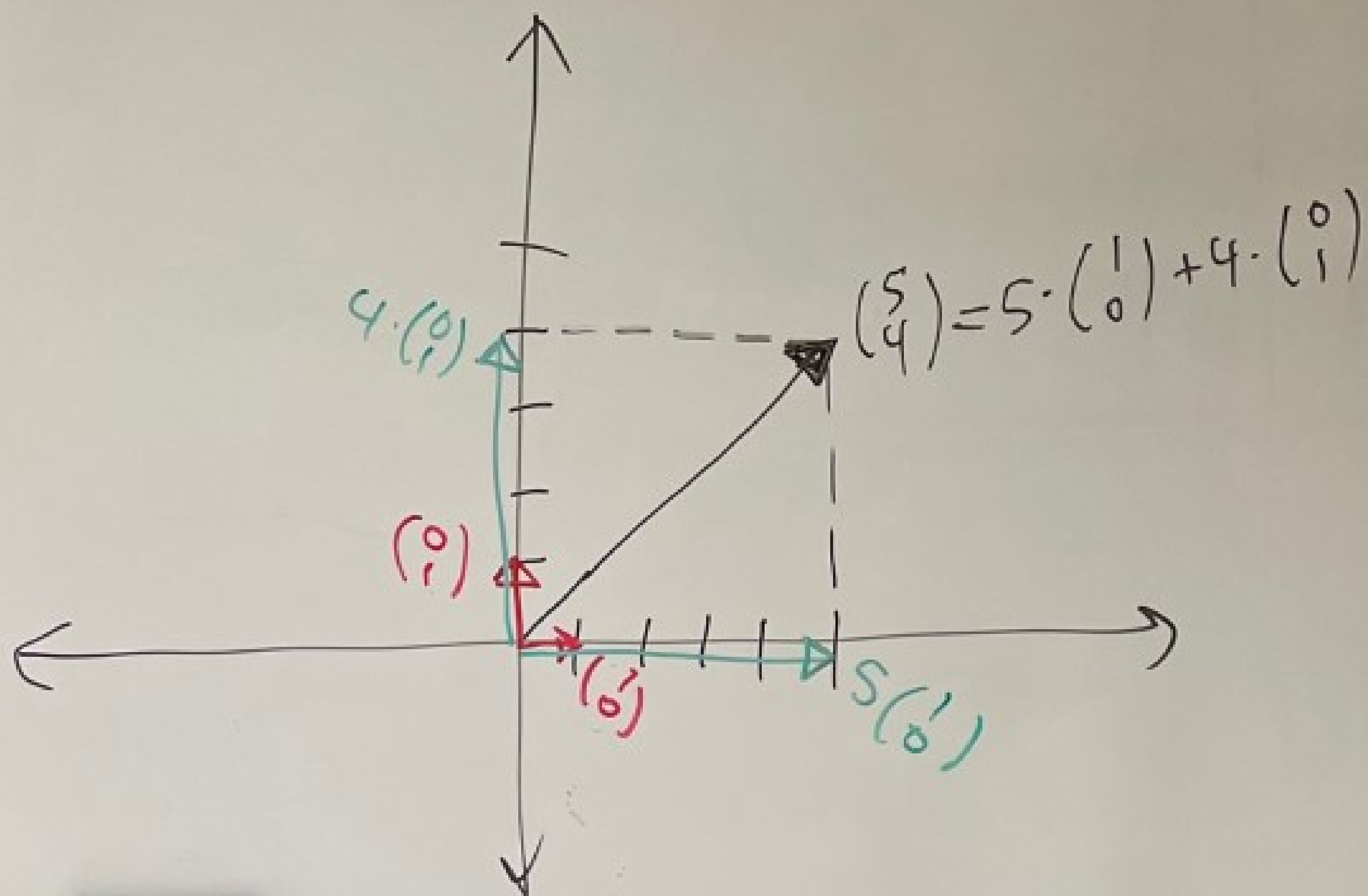
β creates this red axis system

$\gamma = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$ is the standard basis for $\mathbb{R}^2$

$v = \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 5 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So, $[v]_\gamma = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

The standard basis makes the usual xy-axis system



Ex: Let $V = P_2(\mathbb{R})$, $F = \mathbb{R}$.

Let $\beta = [1, 1+x, 1+x+x^2]$

You can show β is a basis for $P_2(\mathbb{R})$.
I think we did in class sometime ago? (or its HW)

Pick $v = 2 - x + 3x^2$.

Let's find $[v]_\beta$.

We need to solve

$$2-x+3x^2 = c_1(1) + c_2(1+x) + c_3(1+x+x^2)$$

$$\underbrace{2-x+3x^2}_{\checkmark}$$

This gives:

$$2-x+3x^2 = (c_1+c_2+c_3) + (c_2+c_3)x + c_3x^2$$

Equate coefficients to get:

$$\boxed{\begin{aligned} ①\quad & c_1+c_2+c_3 = 2 \\ ②\quad & c_2+c_3 = -1 \\ ③\quad & c_3 = 3 \end{aligned}}$$

already in reduced form

$$\boxed{\begin{aligned} ③\quad & c_3 = 3 \\ ②\quad & c_2 = -1-c_3 = -1-3 = -4 \\ ①\quad & c_1 = 2-c_2-c_3 = 2-(-4)-3 = 3 \end{aligned}}$$

$$\boxed{2-x+3x^2 = 3(1) - 4(1+x) + 3(1+x+x^2)}$$

$$\boxed{[2-x+3x^2]_\beta = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}}$$

$$\gamma = [1, x, x^2] \leftarrow \text{standard basis}$$

$$[2 - x + 3x^2]_\gamma = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Def: Let $T: V \to W$ be a linear transformation between two finite-dimensional vector spaces V and W over a field $\mathbb{F}$.

Let $\beta = [v_1, v_2, \dots, v_n]$ be an ordered basis for V

Let γ be an ordered basis for W.

The matrix

$$[T]_\beta^\gamma = \left(\underbrace{[T(v_1)]_\gamma}_{\text{first column}} \;\Big|\; \underbrace{[T(v_2)]_\gamma}_{\text{2nd column}} \;\Big|\; \cdots \;\Big|\; \underbrace{[T(v_n)]_\gamma}_{\text{n-th column}} \right)$$

is called the <u>matrix of T with respect to β and γ</u>

If $V = W$ and $\beta = \gamma$ then we just write $[T]_\beta$ instead of $[T]_\beta^\beta$