Last time

$$V \xrightarrow{T} W$$

ordered basis $\gamma = [v_1, v_2, \dots, v_n]$

ordered basis β

If $V = W$, $\beta = \gamma$, then

$$[T]_\beta^\beta = [T]_\beta$$

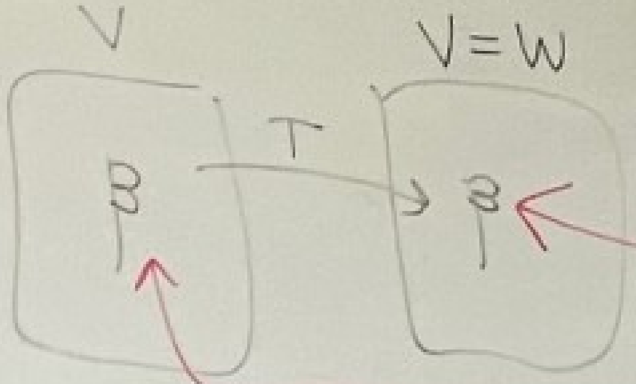
$$[T]_\gamma^\beta = \left([T(v_1)]_\beta \,\middle|\, [T(v_2)]_\beta \,\middle|\, \cdots \,\middle|\, [T(v_n)]_\beta \right)$$

Ex: Let $V = W = \mathbb{R}^2$ and $F = \mathbb{R}$

Let $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix}$

Let $\beta = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$

Let's calculate $[T]_\beta = [T]_\beta^\beta$



get the β coordinates

$$\Rightarrow \quad \begin{aligned} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1+0 \\ 2-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ T\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0+1 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

plug β into T

$$[T]_\beta = [T]_\beta^\beta = \left(\left[T\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_\beta \middle| \left[T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]_\beta \right) = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

Now pick $\beta' = \left[\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \right]$ ← you can check this is a basis for $\mathbb{R}^2$

Let's calculate $[T]_{\beta'} = [T]_{\beta'}^{\beta'}$

$$\begin{array}{ccc} V & & V=W \\ \beta' & \xrightarrow{T} & \beta' \end{array}$$

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1+1\\2-1\end{pmatrix} = \overbrace{\begin{pmatrix}2\\1\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + c\begin{pmatrix}-1\\1\end{pmatrix}}^{\text{write in terms of } \beta'}$$

$$\underbrace{T\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-1+1\\-2-1\end{pmatrix} = \begin{pmatrix}0\\-3\end{pmatrix}}_{\text{plug } \beta' \text{ into } T} = b\begin{pmatrix}1\\1\end{pmatrix} + d\begin{pmatrix}-1\\1\end{pmatrix}$$

Need to solve:

$$\boxed{\begin{aligned} 2 &= a - c \\ 1 &= a + c \end{aligned}}$$

$$\boxed{\begin{aligned} 0 &= b - d \\ -3 &= b + d \end{aligned}}$$

If you solve these You will get

$a = \frac{3}{2}$, $c = -\frac{1}{2}$

$b = \frac{-3}{2}$, $d = -\frac{3}{2}$

So we have:

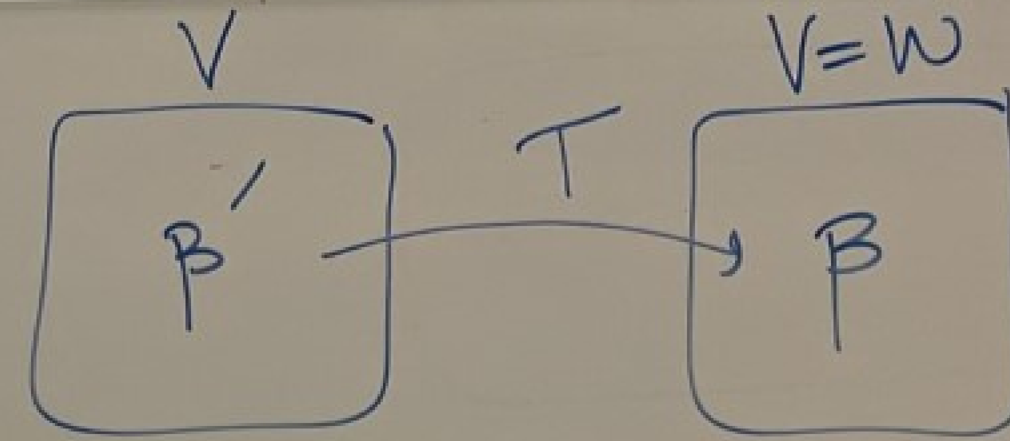
$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} = \frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}$$

$$T\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}0\\-3\end{pmatrix} = \frac{-3}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{3}{2}\begin{pmatrix}-1\\1\end{pmatrix}$$

Thus,

$$[T]_{\beta'} = \left(\left[T\begin{pmatrix}1\\1\end{pmatrix}\right]_{\beta'} \middle| \left[T\begin{pmatrix}-1\\1\end{pmatrix}\right]_{\beta'} \right) = \begin{pmatrix} 3/2 & -3/2 \\ -1/2 & -3/2 \end{pmatrix}$$

Let's calculate $[T]_{\beta'}^{\beta}$



$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} = 2\cdot\begin{pmatrix}1\\0\end{pmatrix} + 1\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

$$\underbrace{T\begin{pmatrix}-1\\1\end{pmatrix} =}_{\text{plug } \beta' \text{ into } T} \underbrace{\begin{pmatrix}0\\-3\end{pmatrix} = 0\cdot\begin{pmatrix}1\\0\end{pmatrix} + (-3)\begin{pmatrix}0\\1\end{pmatrix}}_{\text{write in terms of } \beta}$$

$$[T]_{\beta'}^{\beta} = \left(\left[T\begin{pmatrix}1\\1\end{pmatrix}\right]_{\beta} \middle| \left[T\begin{pmatrix}-1\\1\end{pmatrix}\right]_{\beta} \right) = \begin{pmatrix}2 & 0\\1 & -3\end{pmatrix}$$

What do these matrices do?

Pick $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$[v]_\beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \leftarrow \boxed{v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$[v]_{\beta'} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} \leftarrow \boxed{v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \tfrac{3}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

To find these you'd have to solve this:

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

OR $\begin{array}{l} 1 = a - b \\ 2 = a + b \end{array} \Rightarrow \begin{array}{l} a = 3/2 \\ b = 1/2 \end{array}$

$$[T(v)]_\beta = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$T(v) = T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 2-2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$[T(v)]_{\beta'} = \begin{pmatrix} 3/2 \\ -3/2 \end{pmatrix}$$

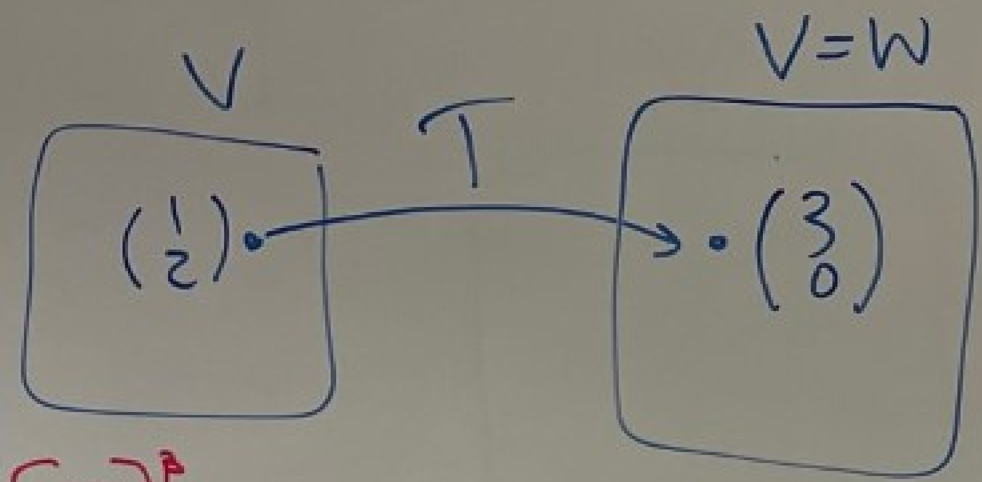
$$T(v) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = a \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

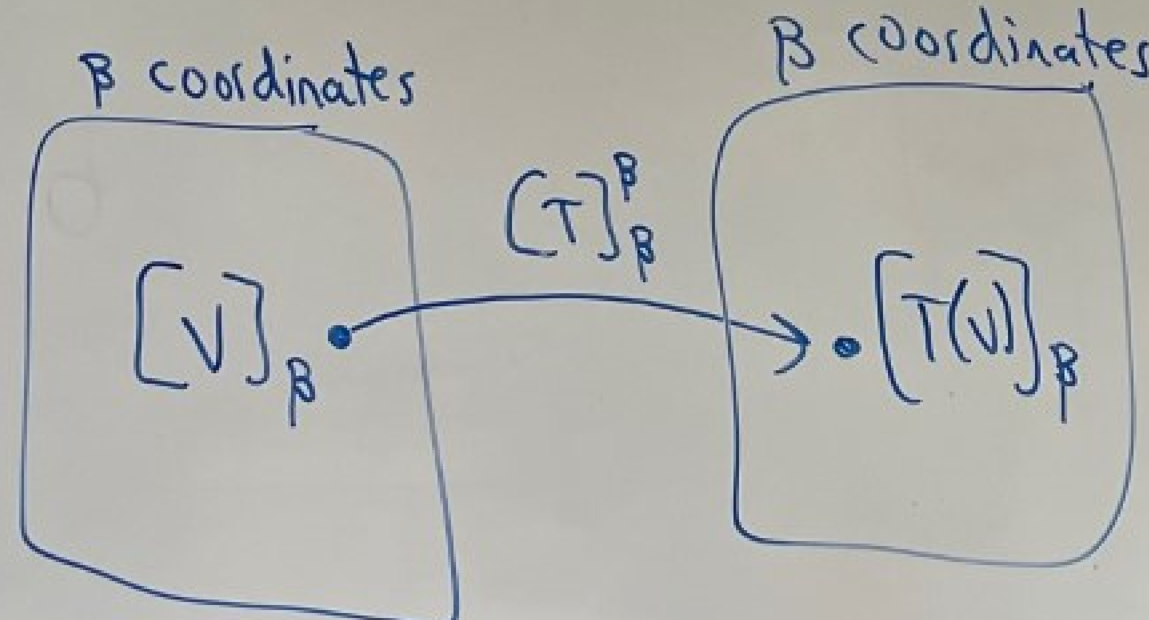
Need to solve: $\boxed{\begin{matrix} 3 = a - b \\ 0 = a + b \end{matrix}} \Rightarrow \begin{matrix} b = -\frac{3}{2} \\ a = \frac{3}{2} \end{matrix}$

$$T(v) = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \frac{3}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\boxed{\text{What does } [T]_\beta = [T]_\beta^\beta \text{ do?}}$$



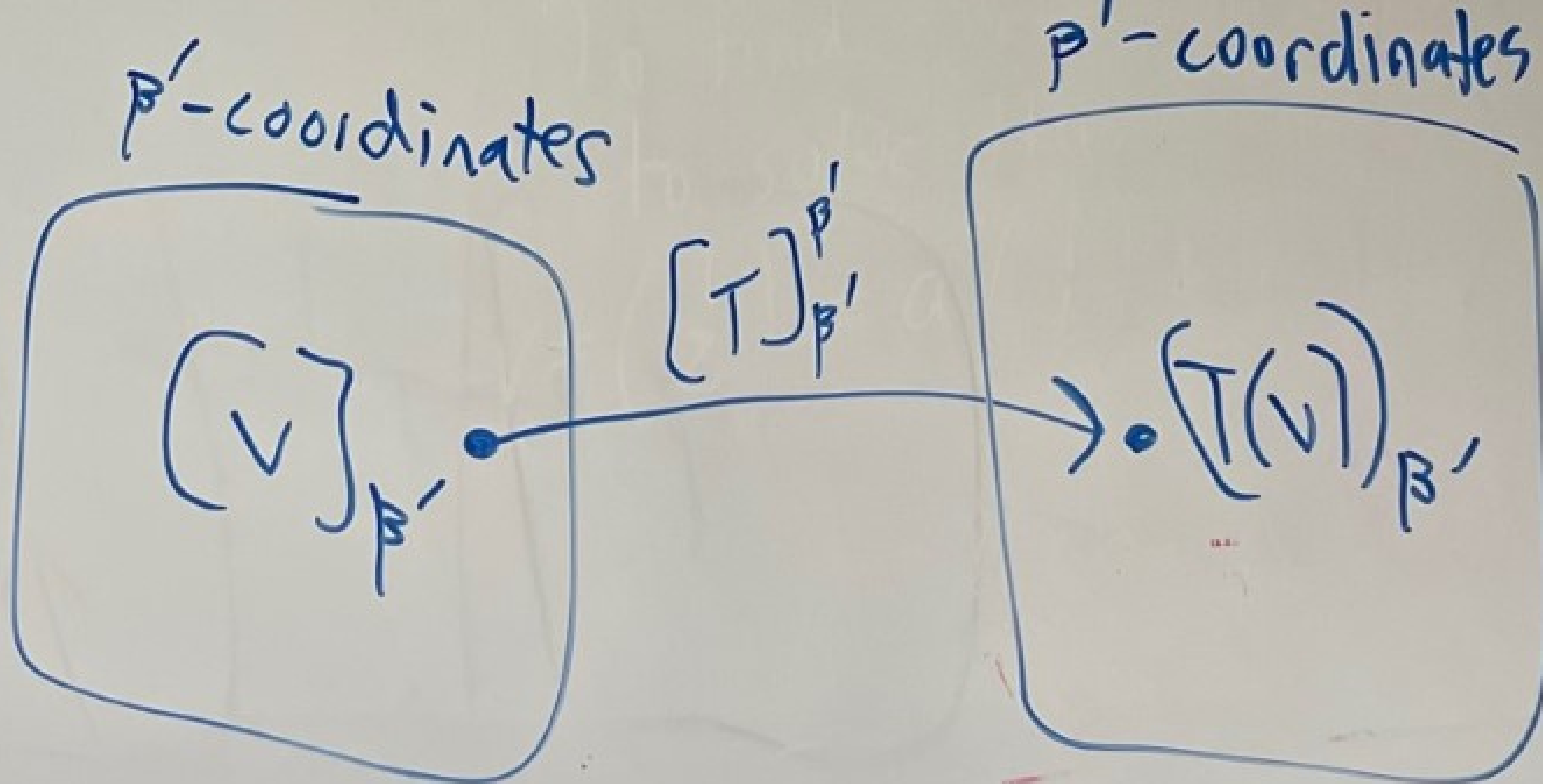
$$\overbrace{[T]_\beta}^{[T]_\beta^\beta}[v]_\beta = \begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1+2\\ 2-2\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix} = [T(v)]_\beta$$



What about $[T]_{\beta'} = [T]_{\beta'}^{\beta'}$?

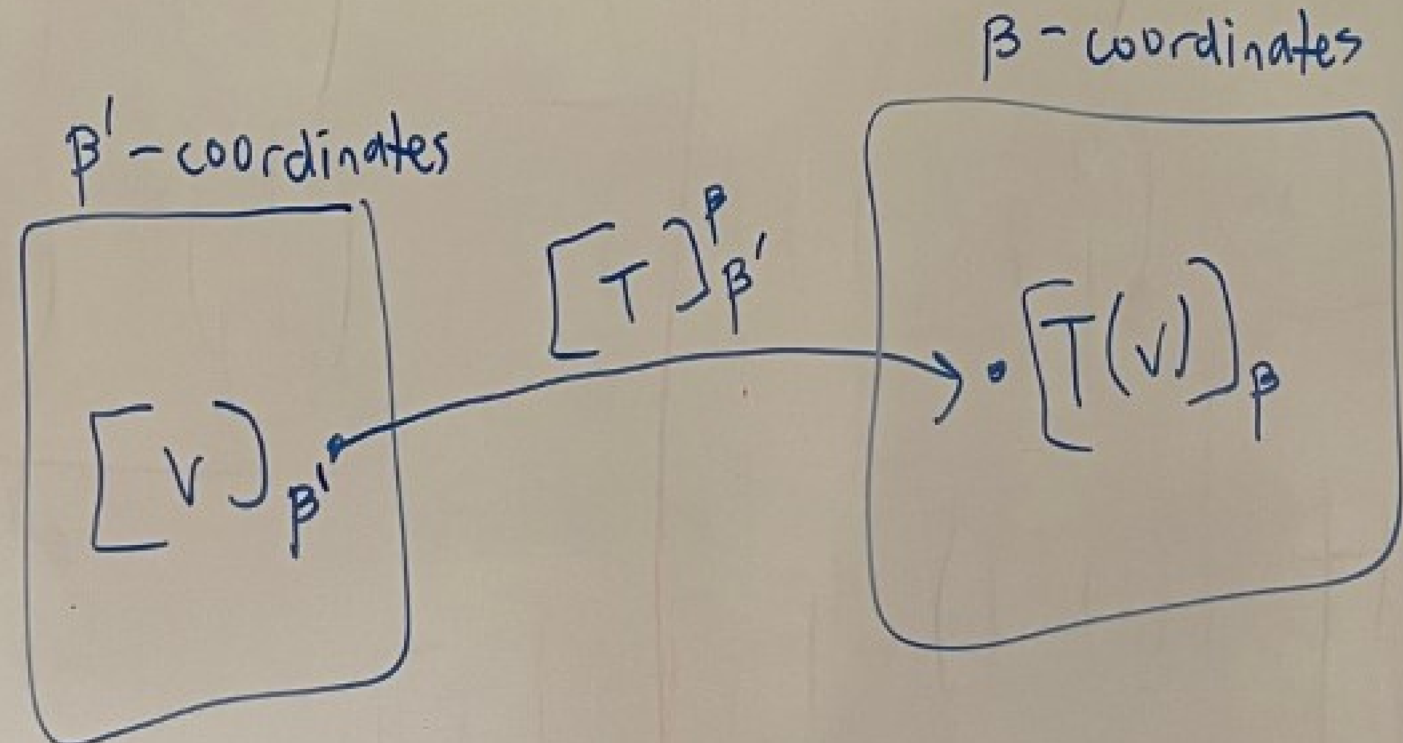
$$[T]_{\beta'}[v]_{\beta'} = \begin{pmatrix} 3/2 & -3/2 \\ -1/2 & -3/2 \end{pmatrix} \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 9/4 - 3/4 \\ -3/4 - 3/4 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -3/2 \end{pmatrix} = [T(v)]_{\beta'}$$



What about $[T]_{\beta'}^{\beta}$?

$$[T]_{\beta'}^{\beta}[v]_{\beta'} = \begin{pmatrix} 2 & 0 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 3+0 \\ 3/2 - 3/2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = [T(v)]_{\beta}$$



Theorem: Let V and W be finite-dimensional vector spaces over a field F.

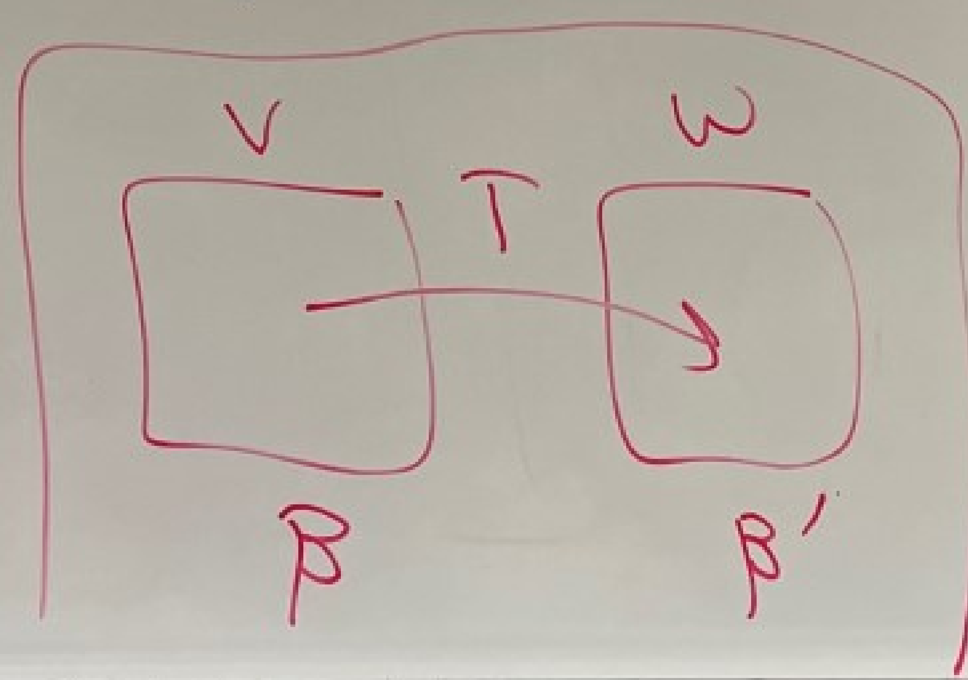
Let $T: V \to W$ be a linear transformation.

Let β be an ordered basis for V

Let β' be an ordered basis for W.

Then,

$$[T]_{\beta}^{\beta'} [v]_{\beta} = [T(v)]_{\beta'} \quad \text{for all } v \in V.$$



proof: Let $\beta = [v_1, v_2, \dots, v_n]$ and $\beta' = [w_1, w_2, \dots, w_m]$.

Let $v \in V$.

Write

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

where $\alpha_1, \alpha_2, \dots, \alpha_n \in F$.

So,

$$[v]_\beta = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Now let's find $[T]_\beta^{\beta'}$.

We have

$$T(v_1) = a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m$$
$$T(v_2) = a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m$$
$$\vdots$$
$$T(v_n) = a_{1n} w_1 + a_{2n} w_2 + \dots + a_{mn} w_m$$

where $a_{ij} \in F$.

So,

$$[T]_\beta^{\beta'} = \left([T(v_1)]_{\beta'} \,\middle|\, [T(v_2)]_{\beta'} \,\middle|\, \cdots \,\middle|\, [T(v_n)]_{\beta'} \right) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

We have

$$T(v) = T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$$

T is linear

$$\overset{\downarrow}{=} \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n)$$

$$= \alpha_1 (a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m)$$

$$+ \alpha_2 (a_{12} w_1 + a_{22} w_2 + \cdots + a_{m2} w_m)$$

$$\vdots \qquad \vdots$$

$$+ \alpha_n (a_{1n} w_1 + a_{2n} w_2 + \cdots + a_{mn} w_m) =$$

$$
\begin{aligned}
&(\alpha_1 a_{11} + \alpha_2 a_{12} + \cdots + \alpha_n a_{1n}) w_1 \\
&+ (\alpha_1 a_{21} + \alpha_2 a_{22} + \cdots + \alpha_n a_{2n}) w_2 \\
&\qquad \vdots \qquad\qquad\qquad \vdots \\
&+ (\alpha_1 a_{m1} + \alpha_2 a_{m2} + \cdots + \alpha_n a_{mn}) w_m
\end{aligned}
$$

w_m

Thus,

$$[T(v)]_{\beta'} = \begin{pmatrix} \alpha_1 a_{11} + \alpha_2 a_{12} + \cdots + \alpha_n a_{1n} \\ \alpha_1 a_{21} + \alpha_2 a_{22} + \cdots + \alpha_n a_{2n} \\ \vdots \\ \alpha_1 a_{m1} + \alpha_2 a_{m2} + \cdots + \alpha_n a_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = [T]_{\beta}^{\beta'} [v]_{\beta}$$

$\blacksquare$