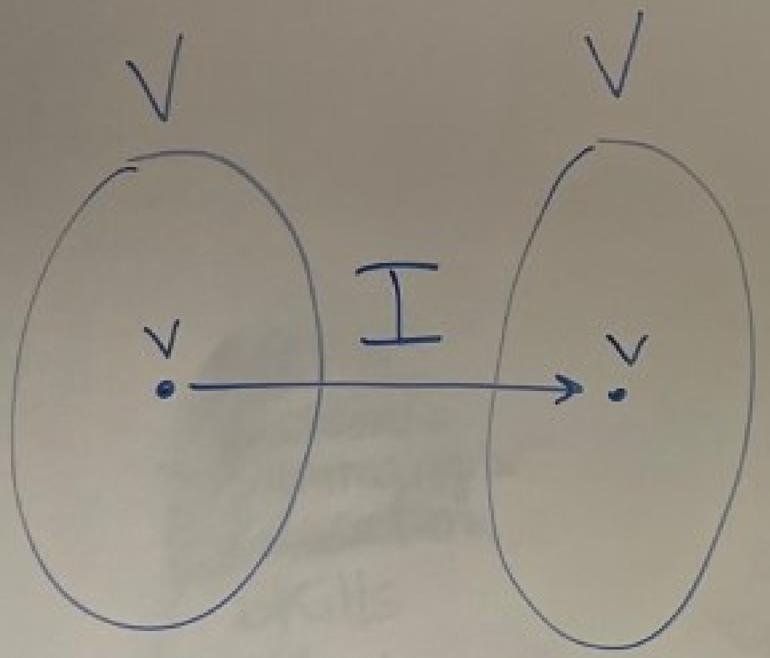



**Theorem:** Let $V$ be a finite-dimensional vector space over a field $F$. Let $\beta$ and $\beta'$ be ordered bases for $V$. Let $I : V \to V$ be the identity linear transformation, that is $I(v) = v$ for all $v$ in $V$.

Then,

$$[I]_{\beta}^{\beta'}[v]_{\beta} = [v]_{\beta'} \qquad \text{for all } v \in V.$$

$[I]_{\beta}^{\beta'}$ is called the change of basis matrix from $\beta$ to $\beta'$

proof: $[I]_{\beta}^{\beta'}[v]_{\beta} = [I(v)]_{\beta'} = [v]_{\beta'}$

thm from last time

Ex: $V = \mathbb{R}^2$, $F = \mathbb{R}$

$$\beta = \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$$\beta' = \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$$

Let's calculate $[I]_{\beta}^{\beta'}$.

$$I\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \end{pmatrix}}^{\text{write answer in terms of } \beta'}$$

$$\underbrace{I\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{plug } \beta \text{ into } I} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} + d\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

We get:

$$\boxed{\begin{matrix} a - b = 1 \\ a + b = 0 \end{matrix}} \Rightarrow \boxed{\begin{matrix} a = \frac{1}{2} \\ b = -\frac{1}{2} \end{matrix}}$$

and

$$\boxed{\begin{matrix} c - d = 0 \\ c + d = 1 \end{matrix}} \Rightarrow \boxed{\begin{matrix} c = 1/2 \\ d = 1/2 \end{matrix}}$$

So,

$$[I]_{\beta}^{\beta'} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

Let's test it!

Let $v = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

What is $[v]_{\beta}$?

$$v = \begin{pmatrix} 2 \\ -3 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 3 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So, $[v]_{\beta} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

Theorem says:

$$[v]_{\beta'} = [I]_{\beta}^{\beta'} [v]_{\beta} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

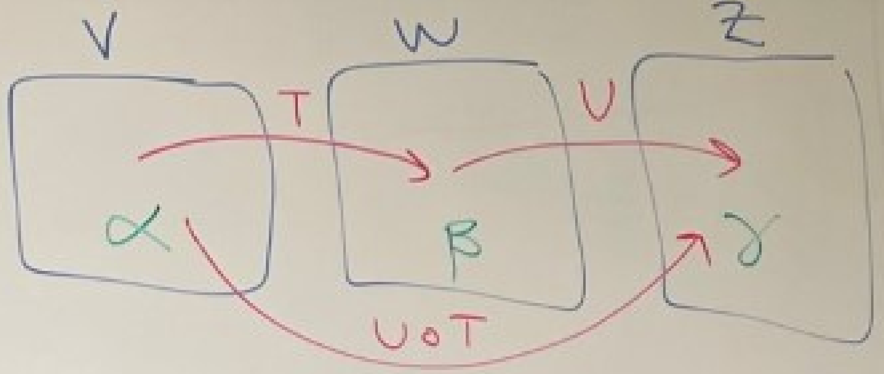
$$= \begin{pmatrix} 1 - 3/2 \\ -1 - 3/2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -5/2 \end{pmatrix}$$

$[v]_{\beta'} = \begin{pmatrix} -1/2 \\ -5/2 \end{pmatrix}$ means that $v = \underbrace{-\tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \tfrac{5}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]}$

And we get:

$$-\tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \tfrac{5}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 + 5/2 \\ -1/2 - 5/2 \end{pmatrix} = \begin{pmatrix} 4/2 \\ -6/2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} = v$$

**HW 4 problem 3** Let $V, W, Z$ be finite-dimensional vector spaces over a field $F$ with ordered bases $\alpha, \beta, \gamma$ respectively. Let $T: V \to W$ and $U: W \to Z$ are linear transformations.




Then, $U \circ T: V \to Z$ is a linear transformation and $[U \circ T]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$

**HW 5 problem 3** Let $V$ be a finite dimensional vector space over a field $F$. Let $\beta$ be an ordered basis for $V$.

Let $I: V \to V$ be the identity linear transformation, that $I(v) = v$ for all $v \in V$.

Let $n = \dim(V)$.

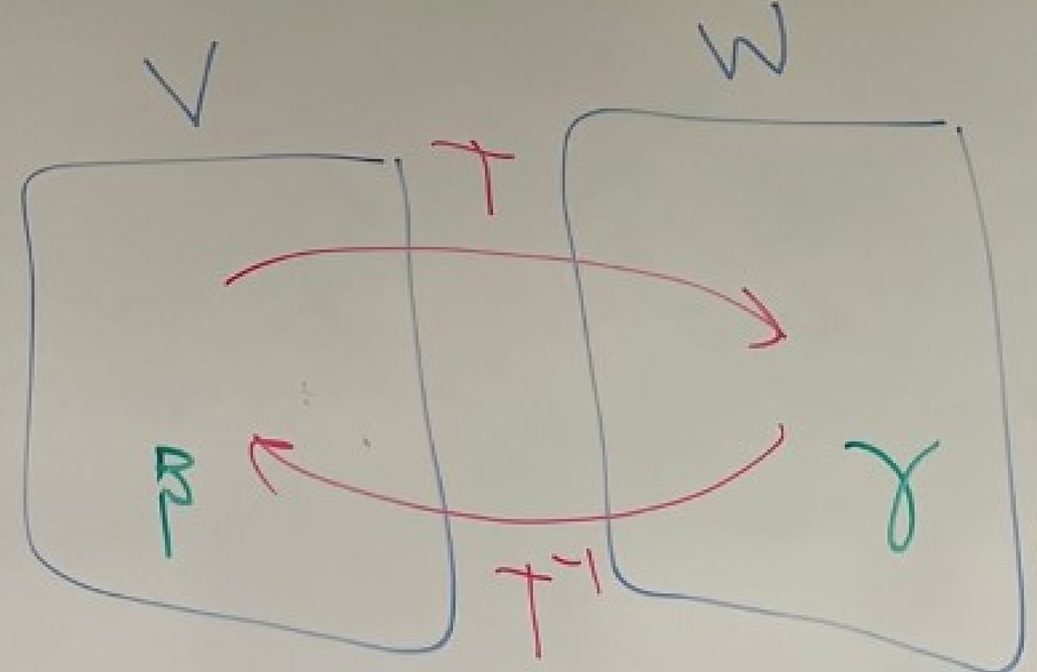
Then,

$$[I]_\beta = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = I_n$$ where $I_n$ is the $n \times n$ identity matrix.

**Theorem:** Let $V$ and $W$ be finite-dimensional vector spaces over a field $F$. Let $T: V \to W$ be a linear transformation. Let $\beta$ and $\gamma$ be ordered basis for $V$ and $W$ respectively.

Then, $T$ is an $\underbrace{\text{isomorphism}}_{\text{one-to-one / onto}}$ iff $[T]_\beta^\gamma$ is invertible.

Furthermore, if this is the case then $[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$




$\boxed{\text{\underline{proof:} See online notes.}}$

**Corollary:** Let $V$ be a finite-dimensional vector space over a field $F$. Let $\beta$ and $\beta'$ be ordered bases for $V$. Let $I: V \to V$ be the identity linear transformation $[I(v) = v \;\; \forall v \in V]$.
Let $Q = [I]_\beta^{\beta'}$ be the change of basis matrix from $\beta$ to $\beta'$.

Then:

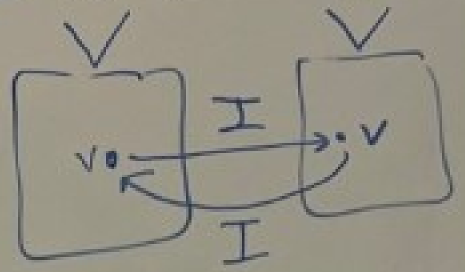
(1) $Q$ is invertible and $Q^{-1} = [I]_{\beta'}^{\beta}$

(2) If $T: V \to V$ is a linear transformation, then

$$\underbrace{[T]_\beta}_{[T]_\beta^\beta} = \underbrace{Q^{-1}[T]_{\beta'}Q}_{[I]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'}[I]_\beta^{\beta'}}$$

**Proof:**

(1) $I$ is invertible and $I^{-1} = I$.




By the previous theorem, $Q = [I]_\beta^{\beta'}$ is invertible and

$$Q^{-1} = [I^{-1}]_{\beta'}^{\beta} = [I]_{\beta'}^{\beta}$$

② We have

$$Q^{-1}[T]_{\beta'}Q \overset{①}{=} [I]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'}[I]_{\beta}^{\beta'}$$

$$= [I]_{\beta'}^{\beta}[T\circ I]_{\beta}^{\beta'}$$

HW 4

#3

$$[U\circ T]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}$$

$$= [I]_{\beta'}^{\beta}[T]_{\beta}^{\beta'}$$

$$(T\circ I)(x) = T(I(x)) = T(x)$$

$$= [I\circ T]_{\beta}^{\beta} = [T]_{\beta}^{\beta} = [T]_{\beta} \qquad \blacksquare$$

$$(I\circ T)(x) = I(T(x)) = T(x)$$