Math 4570

11/28/22

(Recall from last time...)

$T : \mathbb{R}^3 \to \mathbb{R}^3$

$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix}$

characteristic poly

$$f_T(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$
$$= -(\lambda-1)(\lambda-2)^2$$

Eigenvalues

$\lambda = 1 \leftarrow$ algebraic mult. is 1

$\lambda = 2 \leftarrow$ algebraic mult. is 2

$E_1(T)$ has ordered basis $\beta_1 = \left[\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right]$

geometric multiplicity of $\lambda = 1$ is $\dim(E_1(T)) = 1$

Now we continue and calculate an ordered basis for $E_2(T)$.

We have

$$E_2(T) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; \underbrace{T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\substack{T(x) = 2x \\ \lambda = 2}} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; \begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; \begin{pmatrix} -2a & -2c \\ a & +c \\ a & +c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

We need to solve

$$\boxed{\begin{aligned} -2a \quad -2c &= 0 \\ a \quad +c &= 0 \\ a \quad +c &= 0 \end{aligned}}$$

$$\left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{-\frac{1}{2}R_1 \to R_1} \left(\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

got a 1

make these 0

$$\xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This becomes

$$\boxed{\begin{aligned} a \quad +c &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}}$$

leading variables: a

free variables: b, c

Solve for leading variables and give free variables a new name

$$\boxed{\begin{aligned} a &= -c \quad (1) \\ b &= t \quad (2) \\ c &= s \quad (3) \end{aligned}}$$

Back-substitute:

$$\boxed{\begin{aligned} &(3)\ c = s \\ &(2)\ b = t \\ &(1)\ a = -c = -s \end{aligned}}$$

So, $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in E_2(T)$ iff $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -s \\ t \\ s \end{pmatrix}$ where $s, t \in \mathbb{R}$.

$$\text{iff } \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -s \\ 0 \\ s \end{pmatrix} + \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Let $\beta_2 = \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$. From above β_2 spans $E_2(T)$.

By HW 2 #5 since $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are not multiples of each other they form a linearly independent set.

So, β_2 is a basis for $E_2(T)$.

Summary:

Eigenvalues	$\lambda = 1$	$\lambda = 2$
algebraic multiplicity	1	2
basis for $E_\lambda(T)$	$\beta_1 = \left[\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right]$	$\beta_2 = \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$
geometric multiplicity	1	2

$\dim(E_1(T))$ $\dim(E_2(T))$

Note that algebraic multiplicity of λ
= geometric multiplicity of λ

for both λ's.

This will allow us to diagonalize T.

Let

$$\beta = \beta_1 \cup \beta_2 = \left[\underbrace{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}_{\beta_1}, \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\beta_2} \right]$$

We will prove a theorem later that in this situation β is a basis for $\mathbb{R}^3$.

I claim that β will diagonalize T.

We need to compute $[T]_\beta$.

$$T\begin{pmatrix}-2\\1\\1\end{pmatrix} = 1\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix} = 1\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix} + 0\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} + 0\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$T\begin{pmatrix}-1\\0\\1\end{pmatrix} = 2\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} = 0\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix} + 2\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} + 0\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\underbrace{T\begin{pmatrix}0\\1\\0\end{pmatrix} = 2\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}}_{\text{plug } \beta \text{ into } T} = \underbrace{0\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix} + 0\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} + 2\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}}_{\text{write answer in terms of } \beta}$$

So, $$[T]_\beta = \begin{pmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{pmatrix}$$

We diagonalized T!

$\boxed{F = \mathbb{R}}$

<u>Ex:</u> Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$

where $T(f) = f'$. That is,

$T(a + bx + cx^2) = b + 2cx$.

We know from before that T is linear.

Let's find the eigenvalues of T.

Let $\gamma = [1, x, x^2]$. $\leftarrow$ standard basis for $P_2(\mathbb{R})$

Let's find $[T]_\gamma$

We have

$$T(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$

$$T(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$

$$T(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$$

So,

$$[T]_\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Then,

$$f_T(\lambda) = \det\left([T]_\gamma - \lambda I_3\right)$$

$$= \det\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

expand on column 1

$$= -\lambda \cdot \begin{vmatrix} -\lambda & 2 \\ 0 & -\lambda \end{vmatrix} - 0 + 0$$

$$\begin{pmatrix} -\lambda & 1 & 2 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{pmatrix}$$

$$= -\lambda\left[(-\lambda)(-\lambda) - (2)(0)\right]$$

$$= -\lambda^3$$

So, $f_T(\lambda) = -\lambda^3 = -(\lambda - 0)^3$

The only eigenvalue is $\lambda = 0$ which has algebraic multiplicity equal to 3.

And,

$$
\begin{aligned}
E_0(T) &= \{a+bx+cx^2 \mid \overbrace{T(a+bx+cx^2) = 0\cdot(a+bx+cx^2)}^{T(v)=0\cdot v}\} \\
&= \{a+bx+cx^2 \mid \underbrace{b+2cx = 0+0x+0x^2}_{\text{gives } b=0,\ c=0}\} \\
&= \{a \mid a \in \mathbb{R}\} \\
&= \{a\cdot 1 \mid a \in \mathbb{R}\} \\
&= \operatorname{span}(\{1\})
\end{aligned}
$$

So, $\beta = [1]$ is a basis for $E_0(T)$.