HW 5 #4

Theorem: Let V be a finite-dimensional vector space over a field F.

Let $T: V \to V$ be a linear transformation.

Let β and γ be ordered bases for V.

Then, $\det([T]_\beta) = \det([T]_\gamma)$.

proof: We have that

$\det(AB) = \det(A)\det(B)$

$$\det([T]_\beta) \overset{\text{previous thm}}{=} \det\Big(\underbrace{[I]_\gamma^\beta [T]_\gamma [I]_\beta^\gamma}_{\text{change of basis matrices}}\Big) = \det([I]_\gamma^\beta)\det([T]_\gamma)\det([I]_\beta^\gamma) = \det([T]_\gamma)\det([I]_\gamma^\beta)\det([I]_\beta^\gamma)$$

$$= \det([T]_\gamma)\det([I]_\gamma^\beta [I]_\beta^\gamma) = \det([T]_\gamma)\det(I_n) = \det([T]_\gamma)\cdot 1 = \det([T]_\gamma)$$

$I: V \to V$
$I(x) = x$

thm: $([I]_\gamma^\beta)^{-1} = [I]_\beta^\gamma$

$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$

$\blacksquare$

The previous theorem makes the next definition well-defined.

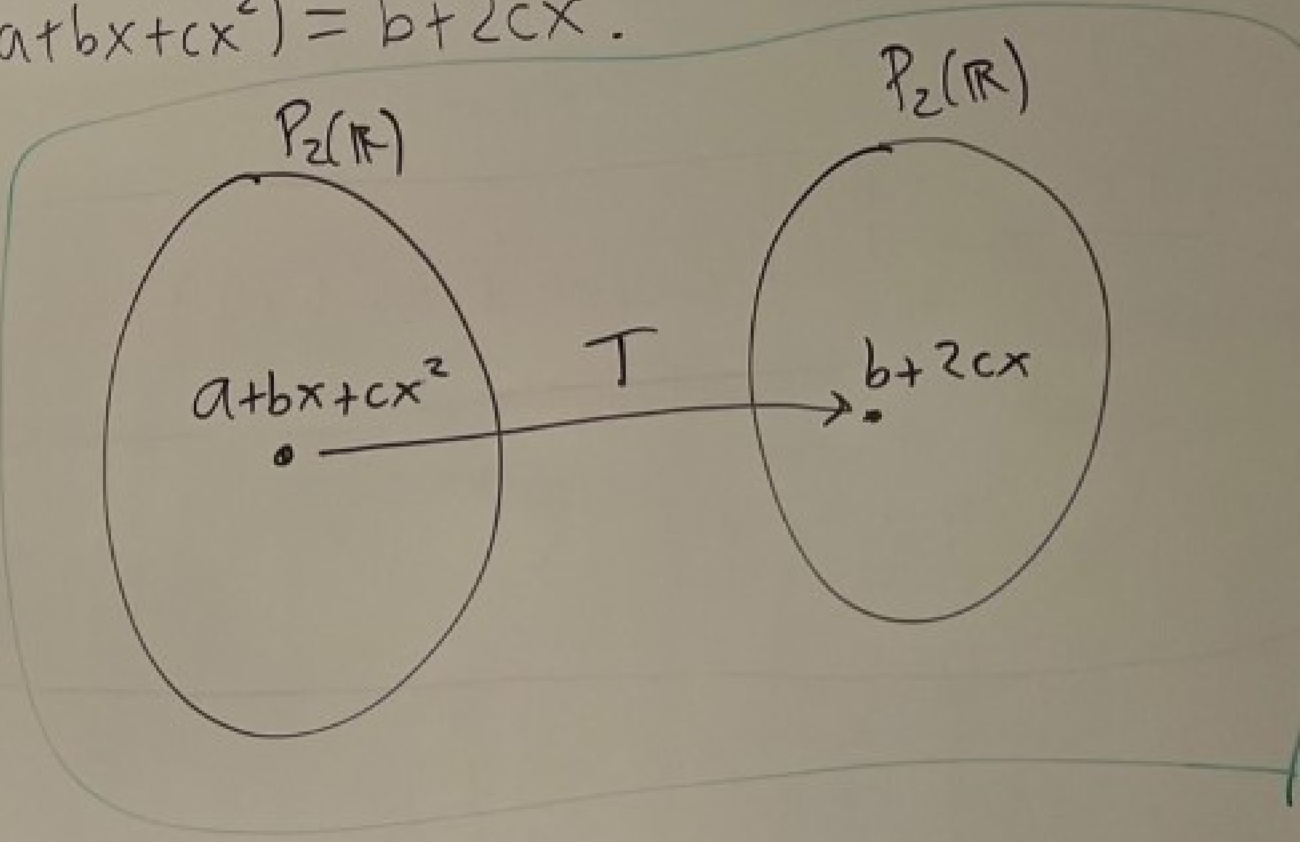
<u>Def:</u> Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation. Define the <u>determinant</u> of T to be

$$\det(T) = \det([T]_\beta)$$

where β is any ordered basis for V.

Ex: Let $V = P_2(\mathbb{R}) = \{a+bx+cx^2 \mid a,b,c \in \mathbb{R}\}$ and $F = \mathbb{R}$.

Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be given by $T(f) = f'$ or $T(a+bx+cx^2) = b+2cx$.



We saw previous that T is linear.

Let's calculate $\det(T)$.

We need a basis for $P_2(\mathbb{R})$.

Let $\beta = [1, x, x^2]$

Let's calculate $[T]_\beta$.

Recall $[T]_\beta = [T]_\beta^\beta$.

$$T(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$
$$T(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$
$$\underbrace{T(x^2) = 2x}_{\text{plug } \beta \text{ into } T} = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$$

$2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$: write answer in terms of β.

So,
$$[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, $\det(T) = \det([T]_\beta)$
$$= \det \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

expand on column I

Note: Let V be a finite-dimensional vector space over a field F.

Let $T : V \to V$ is a linear transformation.

By HW 3 #6(b): T is one-to-one iff T is onto.

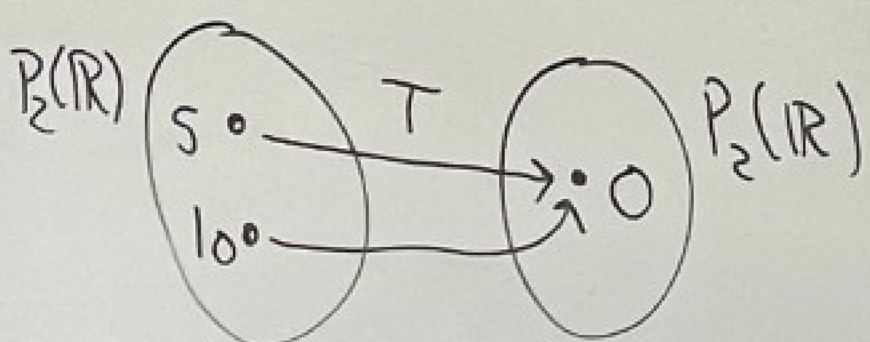
proof uses rank/nullity thm

By HW 5 #5(a): $\det(T) \neq 0$ iff T is invertible.

invertible means one-to-one and onto

So, it makes sense that $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $T(f) = f'$ has $\det(T) = 0$ because T is not one-to-one.

For example $T(5) = 0 = T(10)$.



Theorem: Let V be a finite-dimensional vector space over a field F.

Let $T: V \to V$ be a linear transformation.

Then the following are equivalent:

1. There exists an eigenvector $x \in V$ of T with eigenvalue λ. [Recall $x \neq \vec{0}$]

2. $\det(T - \lambda I) = 0$

3. $N(T - \lambda I) \neq \{\vec{0}\}$

$I: V \to V$ satisfies $I(x) = x \ \forall x \in V$

$(T - \lambda I): V \to V$ is defined as

$$(T-\lambda I)(x) = T(x) - \lambda I(x)$$
$$= T(x) - \lambda x$$

You can show $T - \lambda I$ is a linear transformation.

No

Let

By

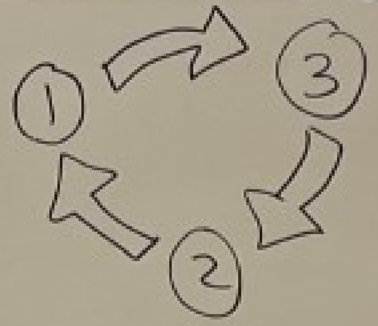
By H

So,

ha

F

Proof: We will prove it like this



(1 $\Rightarrow$ 3) Suppose $x \in V$, $x \neq \vec{0}$, and $T(x) = \lambda x$ for some $\lambda \in F$.

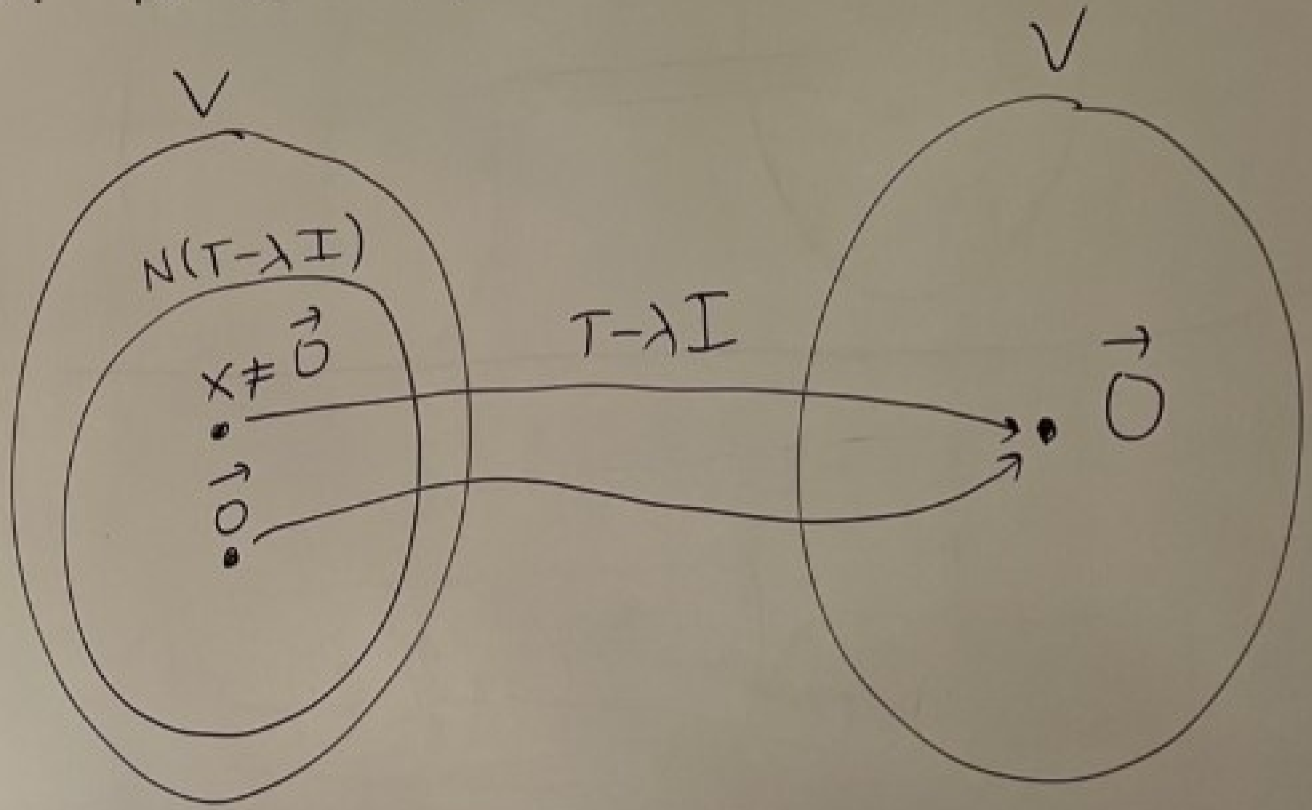
So, $T(x) = \lambda I(x)$.

Thus, $T(x) - \lambda I(x) = \vec{0}$.

Hence, $(T - \lambda I)(x) = \vec{0}$

So, $x \in N(T - \lambda I)$ and $x \neq \vec{0}$.

So, $N(T - \lambda I) \neq \{\vec{0}\}$



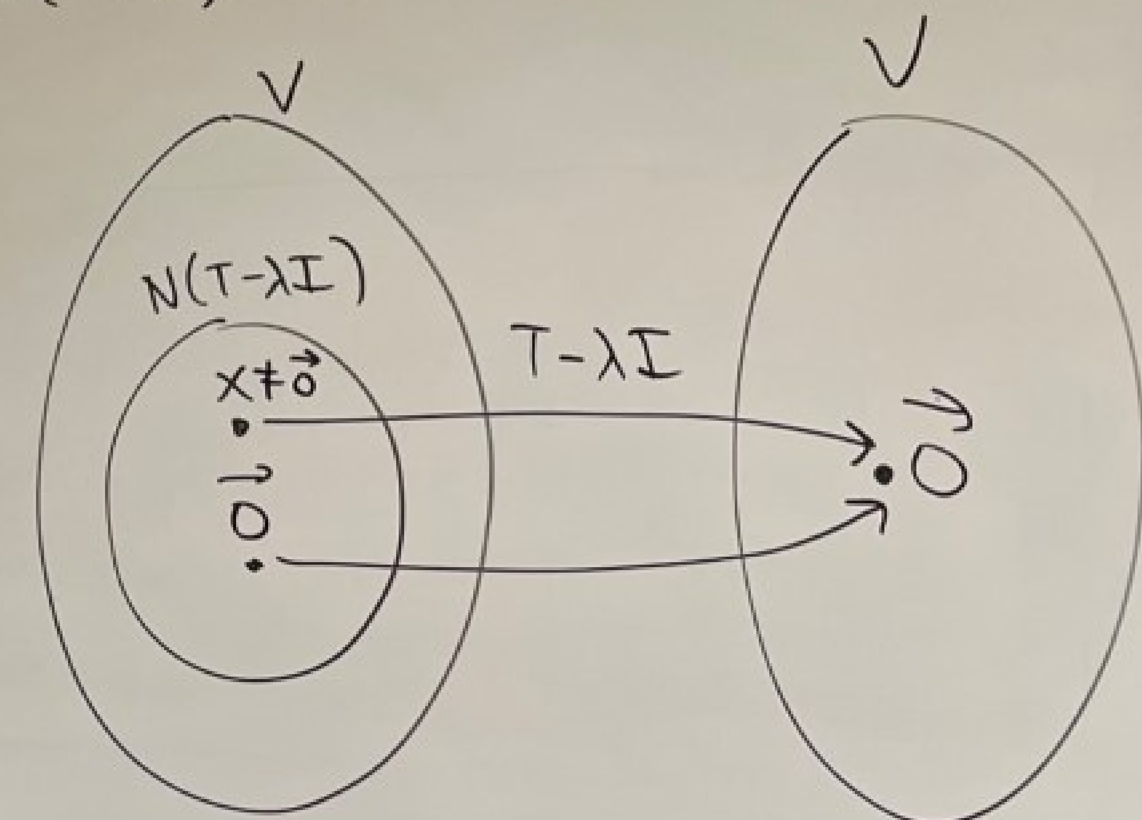
((3 $\Rightarrow$ 2)) Suppose $N(T-\lambda I) \neq \{\vec{0}\}$ for some $\lambda \in F$.

Thus there exists $x \in N(T-\lambda I)$ where $x \neq \vec{0}$.

Thus, $(T-\lambda I)(x) = \vec{0} = (T-\lambda I)(\vec{0})$

Thus, $T-\lambda I$ is not one-to-one

HW 5 #5(a) $\rightarrow$ Ergo, $\det(T-\lambda I) = 0$



《② ⇨ ①》 Suppose $\det(T-\lambda I)=0$ for some $\lambda \in F$.

By HW 5 #5(a) $T-\lambda I$ is not one-to-one.

So there exists $x_1, x_2 \in V$ where

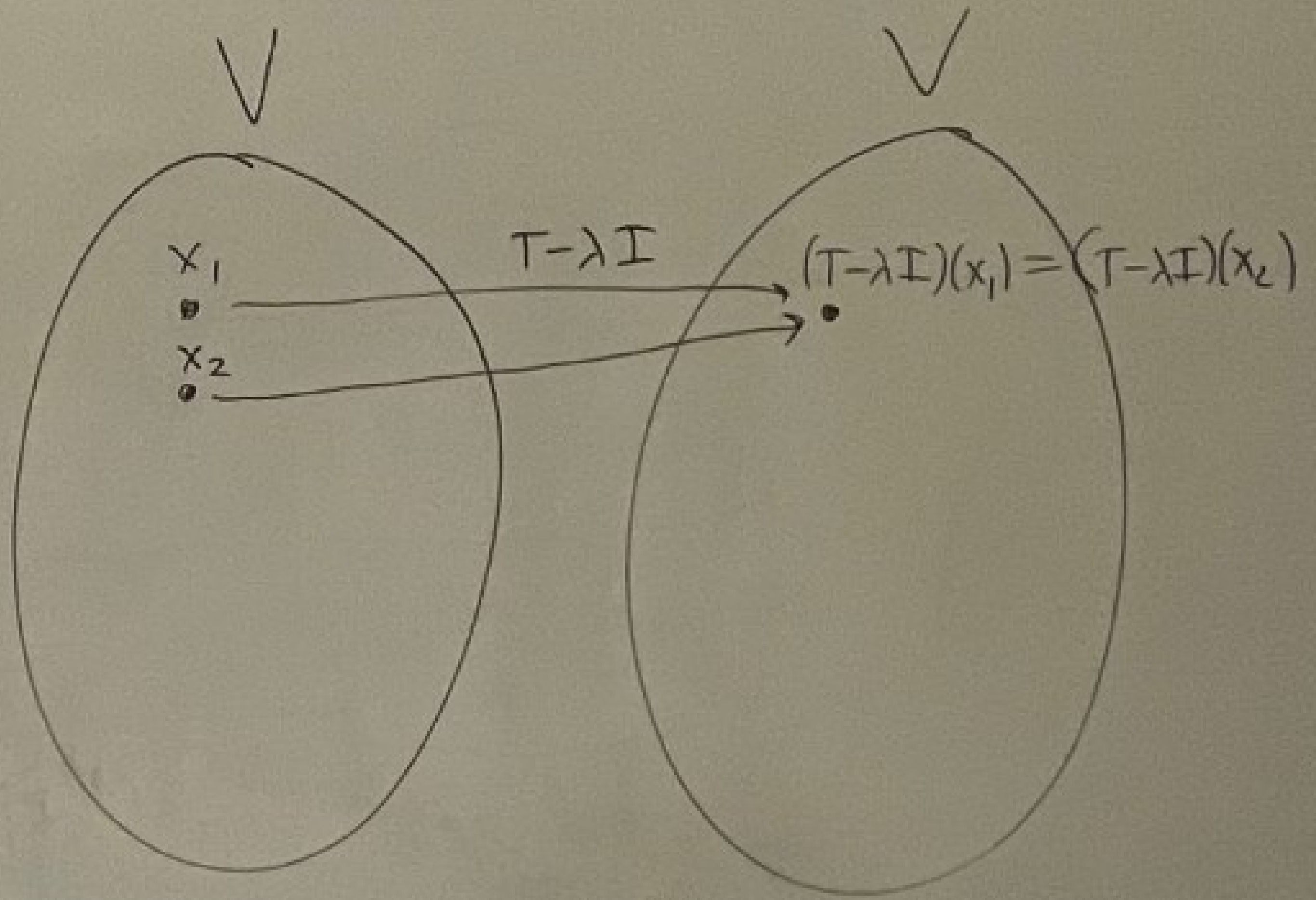
$$(T-\lambda I)(x_1) = (T-\lambda I)(x_2).$$

and $x_1 \neq x_2$.

So, $(T-\lambda I)(x_1) - (T-\lambda I)(x_2) = \vec{0}$.

Since $T-\lambda I$ is a linear transformation we have

$$(T-\lambda I)(x_1 - x_2) = \vec{0}$$



Let $x = x_1 - x_2$. Note $x \neq \vec{0}$ because $x_1 \neq x_2$.

And, $(T - \lambda I)(x) = \vec{0}$

So, $T(x) - \lambda I(x) = \vec{0}$

So, $T(x) = \lambda I(x)$.

Thus, $T(x) = \lambda x$.

So, x is an eigenvector with eigenvalue λ.

<u>Note:</u> $T - \lambda I$ is linear since if $\alpha_1, \alpha_2 \in F$ and $v_1, v_2 \in V$ we have

$$(T-\lambda I)(\alpha_1 v_1 + \alpha_2 v_2) = T(\alpha_1 v_1 + \alpha_2 v_2) - \lambda I(\alpha_1 v_1 + \alpha_2 v_2)$$

$$= \alpha_1 T(v_1) + \alpha_2 T(v_2) - \lambda(\alpha_1 v_1 + \alpha_2 v_2)$$

$$= \alpha_1 T(v_1) - \lambda \alpha_1 v_1 + \alpha_2 T(v_2) - \lambda \alpha_2 v_2 = \alpha_1 T(v_1) - \alpha_1 \lambda I(v_1) + \alpha_2 T(v_2) - \alpha_2 \lambda I(v_2)$$

$$= \alpha_1 [(T-\lambda I)(v_1)] + \alpha_2 [(T-\lambda I)(v_2)]$$