Math 4570

12/7/22

2550 HW 7 - Part 2

(2) $V = M_{2,2}(\mathbb{R})$, $F = \mathbb{R}$

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a+b+c=0,\ a,b,c,d \in \mathbb{R} \right\}$$

Find a basis for W.

Some elements in W:

$\begin{pmatrix} 1 & -1 \\ 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 0 & 10 \\ -10 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ -2 & 6 \end{pmatrix}$, ...

Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W$.

Then $\boxed{a+b+c=0.}$

$\boxed{a=-b-c}$

Then,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -b-c & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} -b & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -c & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$$

$$= b\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So,

$$W = \operatorname{span}\left(\left\{ \underbrace{\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}}_{\text{in } W}, \underbrace{\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}}_{\text{in } W}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{in } W} \right\}\right)$$

Let's show $\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

are linearly independent.

Consider

$$c_1 \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}^{\vec{0}}$$

We need to solve for c_1, c_2, c_3.

The above becomes

$$\begin{pmatrix} -c_1 & c_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -c_2 & 0 \\ c_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which gives

$$\begin{pmatrix} -c_1 - c_2 & c_1 \\ c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This gives
$$\boxed{\begin{aligned} -c_1 - c_2 &= 0 \\ c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \end{aligned}}$$

The only solution is $c_1 = 0, c_2 = 0, c_3 = 0$

Thus, $\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

are a basis for W.

And $\dim(W) = 3$

HW 5

⑥ $T: V \to V$ lin. transformation

λ is an eigenvalue.

$E_\lambda(T) = \{ x \in V \mid T(x) = \lambda x \}$

(a) Show $E_\lambda(T)$ is a subspace of V.

(i) Is $\vec{0} \in E_\lambda(T)$?

We know that $T(\vec{0}) = \vec{0}$ because T is a linear transformation.

Thus, $T(\vec{0}) = \vec{0} = \lambda \cdot \vec{0}$

Hence, $\vec{0} \in E_\lambda(T)$.

(ii) Is $E_\lambda(T)$ closed under addition?

Let $x_1, x_2 \in E_\lambda(T)$.

Then, $T(x_1) = \lambda x_1$ and $T(x_2) = \lambda x_2$.

So,

$$T(x_1+x_2) \overset{\text{Since } T \text{ is linear}}{=} T(x_1)+T(x_2)$$
$$= \lambda x_1 + \lambda x_2$$
$$= \lambda(x_1+x_2)$$

We may conclude that $x_1+x_2 \in E_\lambda(T)$.

(iii) Is $E_\lambda(T)$ closed under scaling?

Let $x \in E_\lambda(T)$ and $\alpha \in F$.

Since $x \in E_\lambda(T)$ we know $T(x)=\lambda x$

Thus, $T(\alpha x) \underset{T \text{ is linear}}{=} \alpha T(x) = \alpha(\lambda x)$
$$= \lambda(\alpha x)$$

So, $\alpha x \in E_\lambda(T)$.

By (i), (ii), (iii), $E_\lambda(T)$ is a subspace of V.

$\boxed{(i)}$

$\boxed{\text{HW 4 \#4}}$ $T_1: V \to W$, $T_2: V \to W$

Show if $[T_1]_\alpha^\beta = [T_2]_\alpha^\beta$, then $T_1 = T_2$.

<u>proof:</u> Suppose $\alpha = [v_1, v_2, \ldots, v_n]$ and $\beta = [w_1, w_2, \ldots, w_m]$ are ordered bases for V and W.

We are given that

$$[T_1]_\alpha^\beta = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = [T_2]_\alpha^\beta$$

$\boxed{\text{info on } T_1(v_2) = T_2(v_2)}$

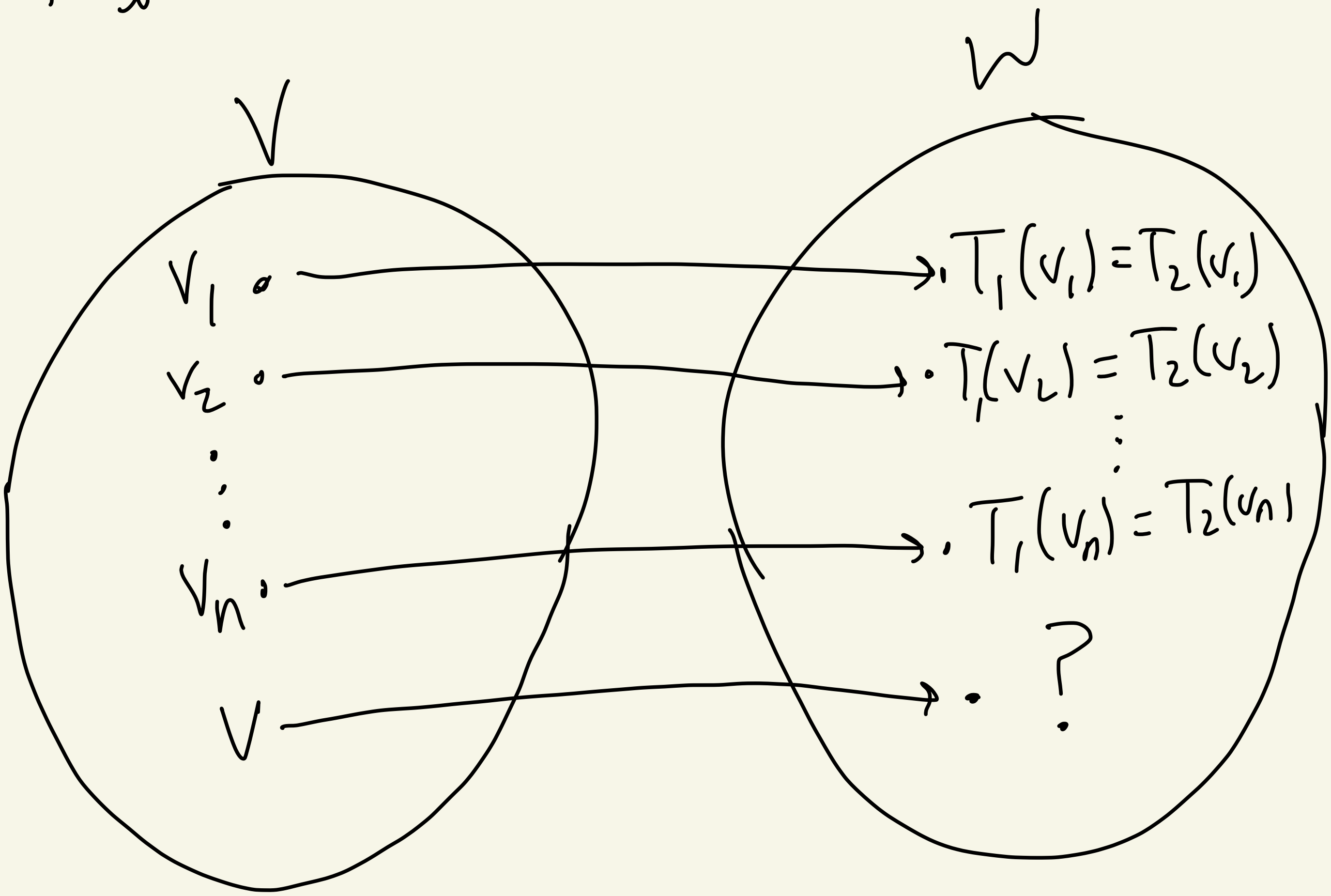
$\boxed{\text{info on } T_1(v_1) \text{ and } T_2(v_1)}$

$\boxed{\begin{array}{l} \alpha = [v_1, v_2, \ldots, v_n] \\ \beta = [w_1, w_2, \ldots, w_m] \end{array}}$

Thus,

$$T_1(v_1) = a_{11}w_1 + a_{21}w_2 + \cdots + a_{m1}w_m = T_2(v_1)$$
$$T_1(v_2) = a_{12}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m = T_2(v_2)$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$T_1(v_n) = a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m = T_2(v_n)$$

So, $T_1(v_i) = T_2(v_i)$ for $i = 1, 2, \dots, n$.



Pick some $v \in V$.

Then, $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ for some $\alpha_1, \dots, \alpha_n \in F$

because $\alpha = [v_1, v_2, \dots, v_n]$ is a basis for V

Then,

$$T_1(v) = T_1(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n)$$

$$\underset{T_1 \text{ is linear}}{=} \alpha_1 T_1(v_1) + \alpha_2 T_1(v_2) + \dots + \alpha_n T_1(v_n)$$

$$= \alpha_1 T_2(v_1) + \alpha_2 T_2(v_2) + \dots + \alpha_n T_2(v_n)$$

$T_1(v_i) = T_2(v_i)$

$$= T_2(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n)$$

T_2 is linear

$$= T_2(v)$$

So, $T_1 = T_2$. $\blacksquare$

HW 3 7(b)

$T: V \to W$

If $\dim(V) > \dim(W)$, then T is not 1-1

Contrapositive

$P \Rightarrow Q$

$\neg Q \Rightarrow \neg P$

Instead prove:

If T is one-to-one, then $\dim(V) \leq \dim(W)$

Proof: Suppose T is one-to-one.

Then, $N(T) = \{0_V\}$

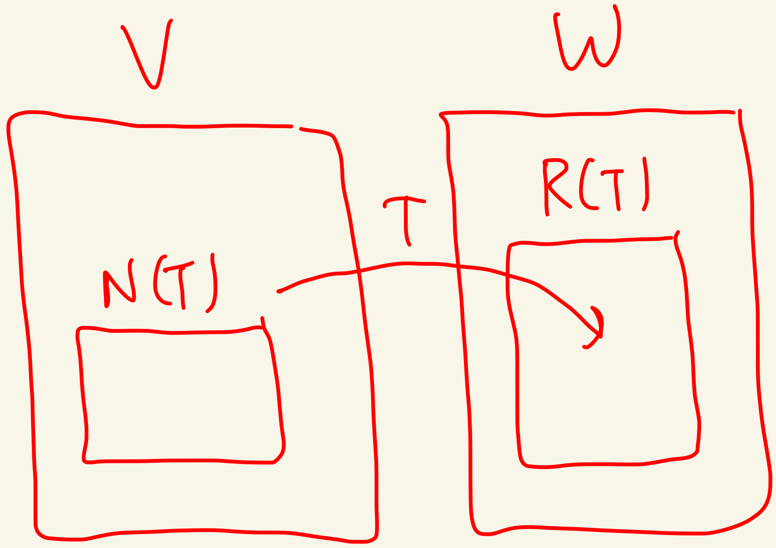
So, $\dim(N(T)) = 0$.

By the rank-nullity theorem

$$\dim(V) = \underbrace{\dim(N(T))}_{0} + \dim(R(T))$$

So, $\dim(V) = \dim(R(T))$.

Since $R(T)$ is a subspace of W we know $\dim(R(T)) \leq \dim(W)$.



Hence, $\dim(V) = \dim(R(T)) \leq \dim(W)$.

So, $\dim(V) \leq \dim(W)$. $\blacksquare$