called the "scalars"

called "vectors"

<u>Def:</u> Let F be a field.

A <u>vector space</u> over F is a set V with two operations. The first operation is addition which takes two elements $v_1, v_2 \in V$ and produces a unique element $v_1 + v_2 \in V$

V is closed under $+$

The second operation is called scalar multiplication which takes $a \in F$ and $v \in V$ and produces a unique element $\underbrace{av}_{\text{you could write } a \cdot v} \in V$.

The following must be true:

(V1) For all $v_1, v_2 \in V$ we have $v_1 + v_2 = v_2 + v_1$. (commutative)

(V2) For all $v_1, v_2, v_3 \in V$ we have $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$ (associative)

(V3) There exists $\vec{0}$ in V where $\vec{0} + v = v + \vec{0} = v$ for all $v \in V$.

[I'm using $\vec{0}$ to distinguish $\vec{0}$ from 0 in F]

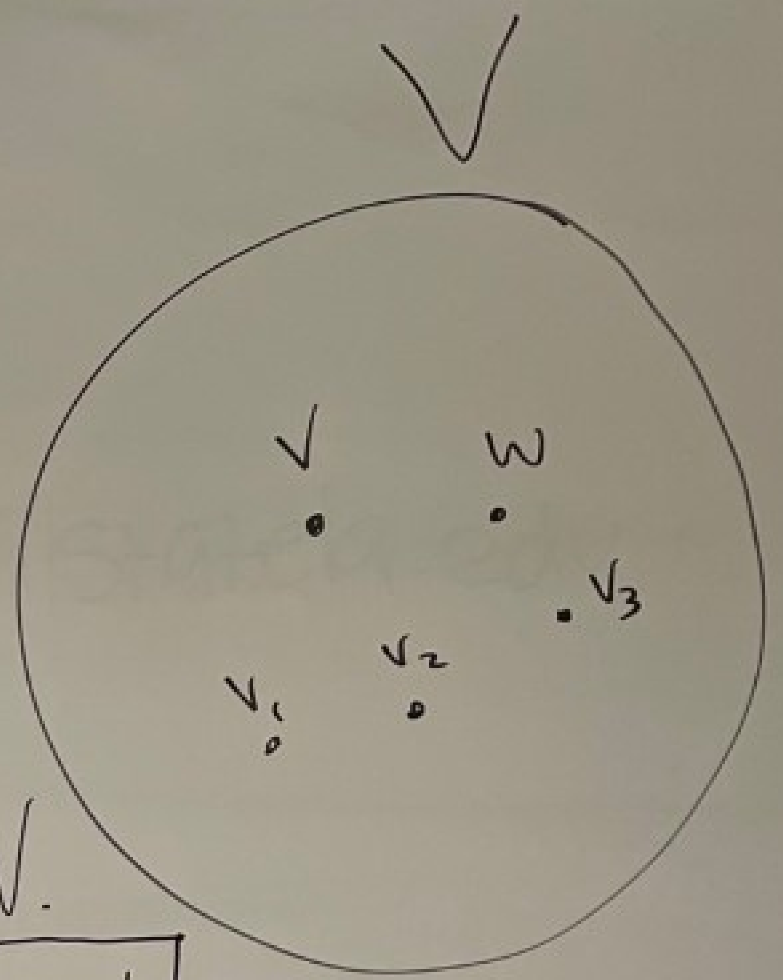
(V4) For every $v \in V$ there exists $w \in V$ where $v + w = w + v = \vec{0}$.

(V5) If 1 is the multiplicative identity of F, then $1v = v$ for all $v \in V$.

(V6) If $a, b \in F$ and $v \in V$, then $(ab)v = a(bv)$.

(V7) If $a \in F$ and $v_1, v_2 \in V$ then $a(v_1 + v_2) = av_1 + av_2$

(V8) If $a, b \in F$ and $v \in V$ then $(a+b)v = av + bv$



<u>Note:</u> It can be shown [HW or later]
that $\vec{0}$ in (V3) is unique and
w in (V4) is unique for each v.

$\vec{0}$ is called the <u>zero vector</u> in V.

The w in (V4) is called the <u>additive inverse</u>
of v and we write $w = -v$.

Ex: Let $F=\mathbb{R}$ and $V=\mathbb{R}^2=\{(x,y) \mid x,y\in\mathbb{R}\}$

Then, $V=\mathbb{R}^2$ is a vector space over $F=\mathbb{R}$

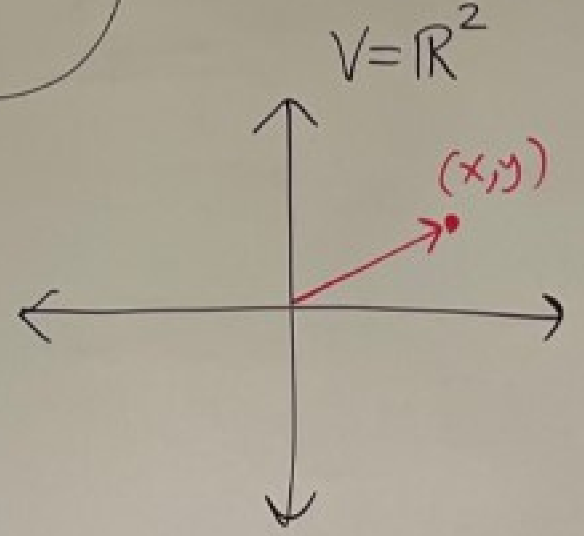
where

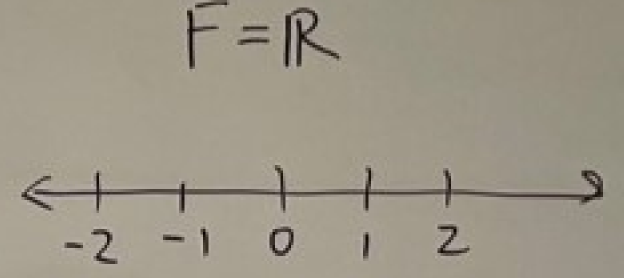
$$(a,b)+(x,y)=(a+x,\ b+y)$$ vector addition

$$c(x,y)=(cx,cy)$$ Scaling a vector

Ex: $(1,7)+(-3,10)=(-2,17)$

$$\tfrac{1}{2}(5,-6)=\left(\tfrac{5}{2},-3\right)$$





Ex: Let F be a field. Let $n \geq 1$.

Let $V = F^n = \{ (a_1, a_2, \dots, a_n) \mid a_1, a_2, \dots, a_n \in F \}$

Then $V = F^n$ is a vector space over F using the following operations. Given

$v = (a_1, a_2, \dots, a_n)$ and $w = (b_1, b_2, \dots, b_n)$ from V

and $\alpha \in F$ define

Generalize $V = \mathbb{R}^2$

$$v + w = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

and $$\alpha v = (\alpha a_1, \alpha a_2, \dots, \alpha a_n)$$

<u>Proof:</u> Let $\alpha, \beta \in F$ and $v, w, z \in V = F^n$ where $v = (v_1, v_2, \dots, v_n)$, $w = (w_1, w_2, \dots, w_n)$ and $z = (z_1, z_2, \dots, z_n)$.

$\alpha \leftarrow$ alpha

$\beta \leftarrow$ beta

(V1) We have

$$v + w \underset{\text{def of } +}{=} (v_1 + w_1, v_2 + w_2, \dots, v_n + w_n) \overset{F2}{=} (w_1 + v_1, w_2 + v_2, \dots, w_n + v_n) = w + v$$

(V2) We have

$$v + (w + z) \overset{\text{def of } +}{=} (v_1, v_2 + \dots + v_n) + (w_1 + z_1, w_2 + z_2, \dots, w_n + z_n)$$

$$\overset{\text{def of } +}{=} \big(v_1 + (w_1 + z_1), v_2 + (w_2 + z_2), \dots, v_n + (w_n + z_n)\big)$$

$$\overset{F2}{=} \big((v_1 + w_1) + z_1, (v_2 + w_2) + z_2, \dots, (v_n + w_n) + z_n\big) =$$

continued on left

$$\overset{\text{def of } +}{=} (v_1+w_1, v_2+w_2, \dots, v_n+w_n) + (z_1, z_2, \dots, z_n) = (v+w)+z$$

(V3) Define $\vec{0} = (0, 0, \dots, 0)$ where 0 is the additive identity of F.

Then $\vec{0} + v = (0+v_1, 0+v_2, \dots, 0+v_n)$

$$\overset{F3}{=} (v_1, v_2, \dots, v_n) = v$$

and $v + \vec{0} = (v_1+0, v_2+0, \dots, v_n+0) \overset{F3}{=} (v_1, v_2, \dots, v_n) = v$

a

(V4) Given $v = (v_1, v_2, \ldots, v_n)$ we have

$-v = (-v_1, -v_2, \ldots, -v_n)$ and

$$v + (-v) = (v_1 + (-v_1), v_2 + (-v_2), \ldots, v_n + (-v_n)) \overset{F4}{=} (0, 0, \ldots, 0)$$

and

$$(-v) + v = (-v_1 + v_1, -v_2 + v_2, \ldots, -v_n + v_n) \underset{F4}{=} (0, 0, \ldots, 0)$$

(V5) Let 1 be the multiplicative identity of F.
Then,

$$1v = 1(v_1, v_2, \dots, v_n) = (1v_1, 1v_2, \dots, 1v_n)$$

$$\underset{F3}{=} (v_1, v_2, \dots, v_n) = v$$

(V6) We have that

$$(\alpha\beta)v = (\alpha\beta)(v_1, v_2, \dots, v_n) = \big((\alpha\beta)v_1, (\alpha\beta)v_2, \dots, (\alpha\beta)v_n\big)$$

$$\overset{F2}{=} \big(\alpha(\beta v_1), \alpha(\beta v_2), \dots \alpha(\beta v_n)\big)$$

$$= \alpha(\beta v_1, \beta v_2, \dots, \beta v_n)$$

$$= \alpha\big[\beta(v_1, v_2, \dots, v_n)\big] = \alpha(\beta v)$$

(V7) We have

$$\alpha(v+w) = \alpha(v_1+w_1, v_2+w_2, \dots, v_n+w_n) = (\alpha(v_1+w_1), \alpha(v_2+w_2), \dots, \alpha(v_n+w_n))$$

$$\overset{F2}{=} (\alpha v_1 + \alpha w_1, \alpha v_2 + \alpha w_2, \dots, \alpha v_n + \alpha w_n)$$

$$= (\alpha v_1, \alpha v_2, \dots, \alpha v_n) + (\alpha w_1, \alpha w_2, \dots, \alpha w_n)$$

$$= \alpha(v_1, v_2, \dots, v_n) + \alpha(w_1, w_2, \dots, w_n)$$

$$= \alpha v + \alpha w$$

V8 We have that

$$(\alpha+\beta)v = (\alpha+\beta)(v_1, v_2, \dots, v_n)$$

$$= ((\alpha+\beta)v_1, (\alpha+\beta)v_2, \dots, (\alpha+\beta)v_n)$$

$$\overset{F2}{=} (\alpha v_1 + \beta v_1, \alpha v_2 + \beta v_2, \dots, \alpha v_n + \beta v_n)$$

$$= (\alpha v_1, \alpha v_2, \dots, \alpha v_n) + (\beta v_1, \beta v_2, \dots, \beta v_n)$$

$$= \alpha(v_1, v_2, \dots, v_n) + \beta(v_1, v_2, \dots, v_n)$$

$$= \alpha v + \beta v.$$

Since V1 - V8 are true, $V = F^n$ is a vector space over F. $\blacksquare$