Ex: Let F be a field. Let $V = M_{m,n}(F)$ be the set of $m \times n$ matrices with entries from F.

Then one can show that V is a vector space over F using the following operations.

vector addition (do normal matrix addition)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

scaling a vector (just like normal scaling)

$$\alpha \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{pmatrix}$$

The zero vector is $\vec{0} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

Ex: Let $F=\mathbb{R}$ and

$$V = M_{3,2}(\underset{F=\mathbb{R}}{\mathbb{R}}) = \left\{ \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \;\middle|\; a,b,c,d,e,f \in \underset{F=\mathbb{R}}{\mathbb{R}} \right\}$$

$$\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Examples of adding and scaling

$$\begin{pmatrix} 1 & -1 \\ 1/2 & 0 \\ \pi & \sqrt{2} \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ e & 10 \\ 4/3 & 1/2 \end{pmatrix} = \begin{pmatrix} 1+2 & -1+3 \\ 1/2+e & 0+10 \\ \pi+\frac{4}{3} & \sqrt{2}+\frac{1}{2} \end{pmatrix} = \boxed{\begin{pmatrix} 3 & 2 \\ \frac{1}{2}+e & 10 \\ \pi+\frac{4}{3} & \sqrt{2}+\frac{1}{2} \end{pmatrix}}$$

$$\frac{2}{3}\begin{pmatrix} 1 & 0 \\ 2 & 5 \\ \pi & 10 \end{pmatrix} = \begin{pmatrix} 2/3 & 0 \\ 4/3 & 10/3 \\ \frac{2\pi}{3} & \frac{20}{3} \end{pmatrix}$$

Ex: Let $F=\mathbb{R}$ or $F=\mathbb{C}$.

Let $n \geq 0$ be an integer.

Let

$$V = P_n(F) = \{ \underbrace{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n}_{\text{polynomials of degree } \leq n \text{ with coefficients from } F} \mid a_0, a_1, a_2, \ldots, a_n \in F \}$$

$$\begin{matrix} 3 & 2 \\ \tfrac{1}{2}+e & 10 \\ \pi+\tfrac{4}{3} & \sqrt{2}+\tfrac{1}{2} \end{matrix}$$

One can show that $V = P_n(F)$ is a vector space over F using the following operations:

<u>vector addition:</u>

$$(a_0+a_1x+a_2x^2+\cdots+a_nx^n)+(b_0+b_1x+b_2x^2+\cdots+b_nx^n)$$
$$=(a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2+\cdots+(a_n+b_n)x^n$$

<u>Scaling:</u>

$$\alpha(a_0+a_1x+a_2x^2+\cdots+a_nx^n)=(\alpha a_0)+(\alpha a_1)x+(\alpha a_2)x^2+\cdots+(\alpha a_n)x^n$$

<u>Zero vector:</u>

$$\vec{0}=0+0x+0x^2+\cdots+0x^n$$

x^n

$a_n)x^n$

We say that

$$a_0+a_1x+a_2x^2+\cdots+a_nx^n=b_0+b_1x+b_2x^2+\cdots+b_nx^n$$

if

$$a_0=b_0,\ a_1=b_1,\ a_2=b_2,\ \ldots,\ a_n=b_n$$

defining equality

Ex: Let $F=\mathbb{R}$.

Let $V = P_4(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \mid a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}\}$

$F=\mathbb{R}$ $F=\mathbb{R}$

$$= \{1 - x + \pi x^3,\ x + x^3 + 51x^4,\ \underset{10+0x+0x^2+0x^3+0x^4}{10},\ \overbrace{0+0x+0x^2+0x^3+0x^4}^{\vec{0}},\ \dots\}$$

Examples of adding & scaling:

$$(1 - x + x^3 + 4x^4) + (9 + x - x^2 + 6x^4) = 10 - x^2 + x^3 + 10x^4$$

$$\tfrac{1}{2}(2 + x - 3x^3 + x^4) = 1 + \tfrac{1}{2}x - \tfrac{3}{2}x^3 + \tfrac{1}{2}x^4$$

$P_4(\mathbb{R})$ is like $\mathbb{R}^5$

$$1 + x - x^3 + 2x^4 = 1 + x + 0x^2 - x^3 + 2x^4 \leftarrow P_4(\mathbb{R})$$

$$\updownarrow$$

$$(1, 1, 0, -1, 2) \leftarrow \mathbb{R}^5$$

Theorem: Let V be a vector space over a field F.

1. The element $\vec{0}$ from V3 is unique.
2. Given $v \in V$ the w from V4 where $v + w = w + v = \vec{0}$ is unique.

proof:

1. Suppose there exist $\vec{0}_1$ and $\vec{0}_2$ in V where

$$\vec{0}_1 + v = v + \vec{0}_1 = v \text{ for all } v \in V$$

and $\vec{0}_2 + v = v + \vec{0}_2 = v$ for all $v \in V$.

F.

Then,

$$\vec{O}_1 = \vec{O}_1 + \vec{O}_2 = \vec{O}_2$$

$v = v + \vec{O}_2$

$v = \vec{O}_1$

Use $\vec{O}_2$'s zero vector powers

$\vec{O}_1 + v = v$

$v = \vec{O}_2$

use $\vec{O}_1$'s zero vector powers

So, $\vec{O}_1 = \vec{O}_2$.

There can be only one zero vector.

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② Let $v \in V$.

Suppose we had $w_1, w_2 \in V$ where

$$v + w_1 = w_1 + v = \vec{0}$$

and $v + w_2 = w_2 + v = \vec{0}$.

Let's show that $w_1 = w_2$.

We have $v + w_2 = \vec{0}$

Add w_1 to both sides on the left to get

$$w_1 + (v + w_2) = \underbrace{w_1 + \vec{0}}_{w_1}$$

Using associativity V2 we get $\underbrace{(w_1 + v)}_{w_1 + v = \vec{0}} + w_2 = w_1$.

So we get $\underbrace{\vec{0} + w_2}_{w_2} = w_1$

Thus, $w_2 = w_1$.

So there is only one w
satisfying $v + w = w + v = \vec{0}$ in V4.

∎

Def: Let V be a vector space over a field F.
Let $W \subseteq V$.
We say that W is a <u>subspace</u> of V if W is also a vector space over F using the same vector addition and vector scaling as V uses.