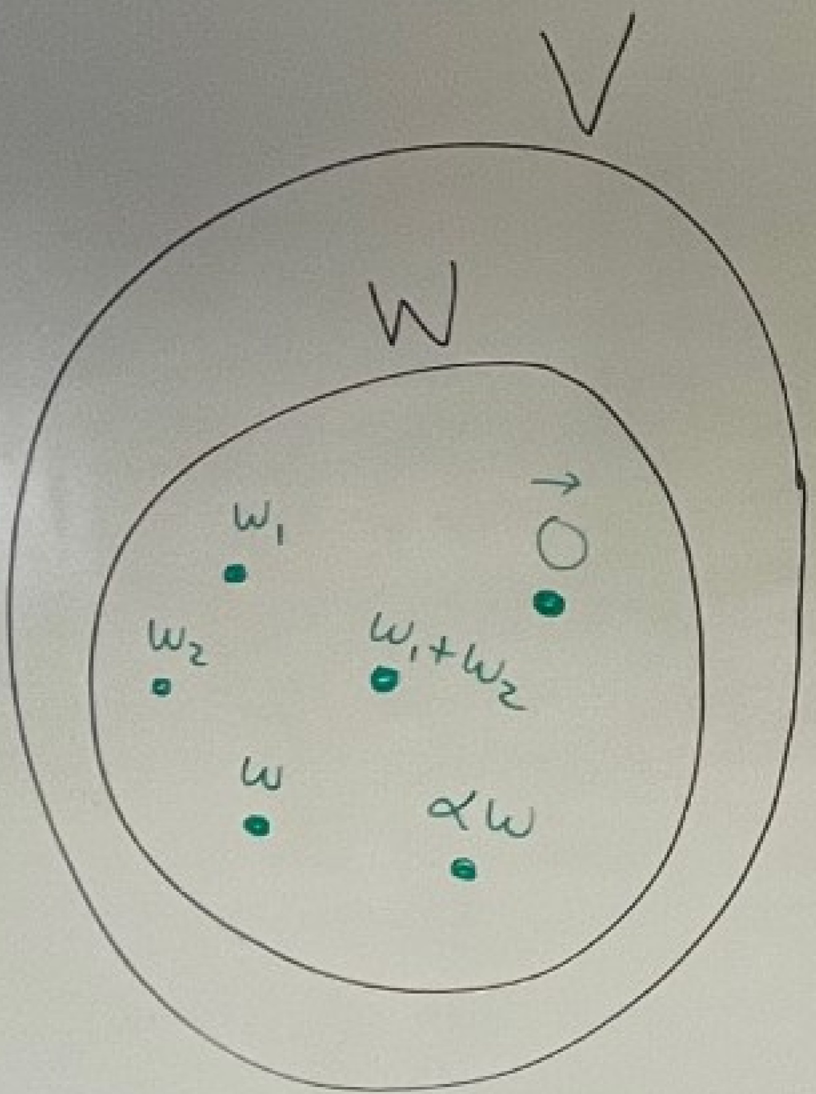


Theorem: Let V be a vector space over a field F.
Let W be a subset of V.
W is a subspace of V iff the following three conditions hold:

1. $\vec{0} \in W$ — you could show $W \neq \emptyset$
2. If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$. — W is closed under $+$
3. If $\alpha \in F$ and $w \in W$, then $\alpha w \in W$ — W is closed under scaling

Proof: In HW. ▨

d F.

HW.

HW 1

(2) (c) Let $V = M_{2,2}(\mathbb{R})$ and $F = \mathbb{R}$.

Let

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{c} a+b+c+d=0 \\ a,b,c,d \in \mathbb{R} \end{array} \right\}$$

Is W a subspace of V?

Some elements of W:

$$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3\pi & \pi \\ \pi & \pi \end{pmatrix}$$

Yes!

Proof that W is a subspace of $V = M_{2,2}(\mathbb{R})$

① $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$ because $0+0+0+0=0$.

② Let $w_1, w_2 \in W$.

Then, $w_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ where $a_1+b_1+c_1+d_1=0$

and $w_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ where $a_2+b_2+c_2+d_2=0$.

Adding $a_1+b_1+c_1+d_1=0$ and $a_2+b_2+c_2+d_2=0$

gives $\boxed{(a_1+a_2)+(b_1+b_2)+(c_1+c_2)+(d_1+d_2)=0.}$ (*)

We have that

$$w_1 + w_2 = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix}.$$

By (*) we know $w_1 + w_2 \in W$.

3 Let $w \in W$ and $\alpha \in \mathbb{R}$ ← $F = \mathbb{R}$

Since $w \in W$ we know

$w = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $\boxed{a+b+c+d=0}$

Then, $\alpha w = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$.

And we have

$$\begin{aligned} \alpha a + \alpha b + \alpha c + \alpha d &= \alpha(a+b+c+d) \\ &= \alpha(0) \\ &= 0. \end{aligned}$$

So, $\alpha w \in W$.

By 1, 2, 3,
we know W is
a subspace of $V = M_{2,2}(\mathbb{R})$. ▨

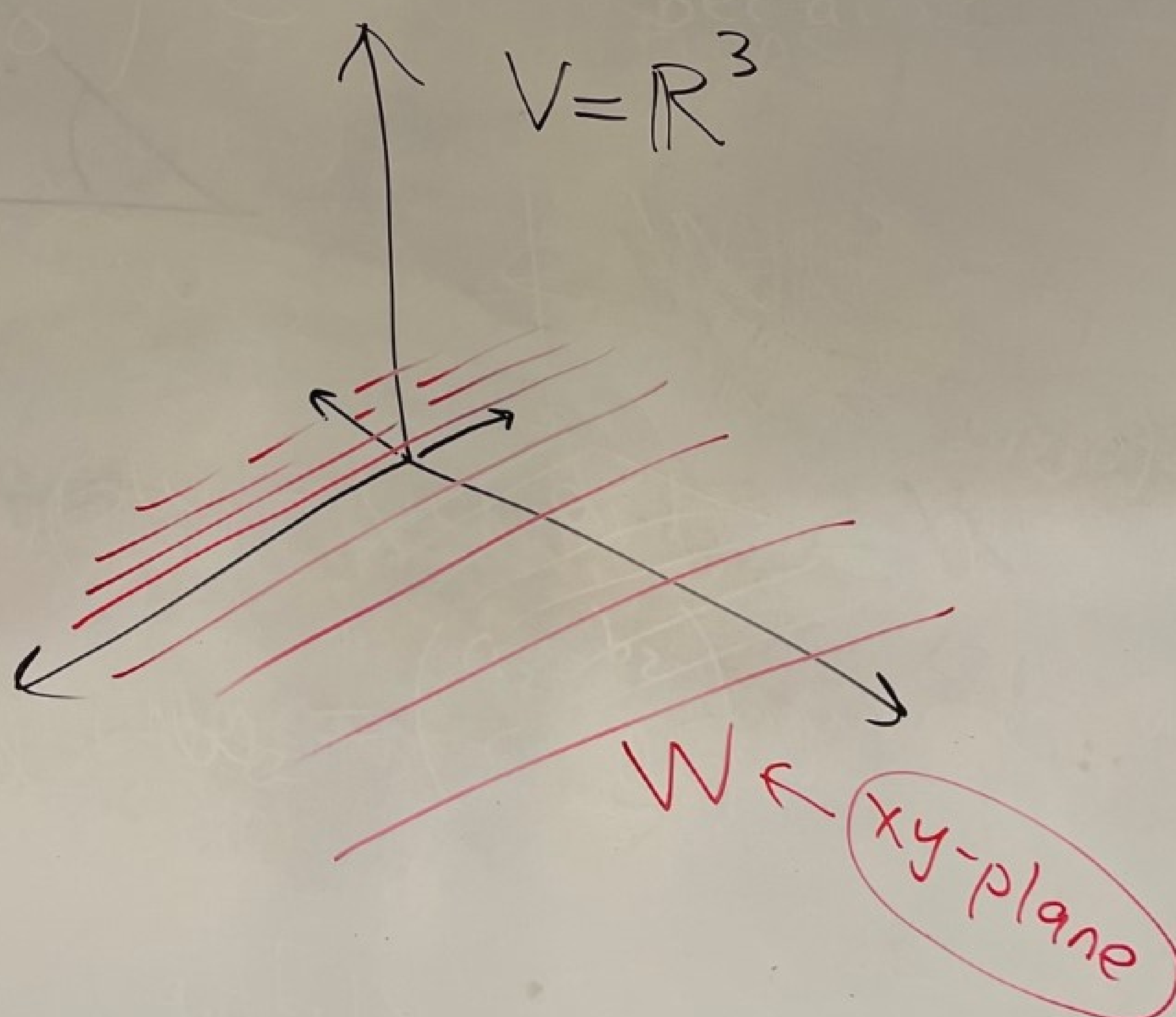
Don't
Walk
here

DANGER!

Ex: $V = \mathbb{R}^3$, $F = \mathbb{R}$

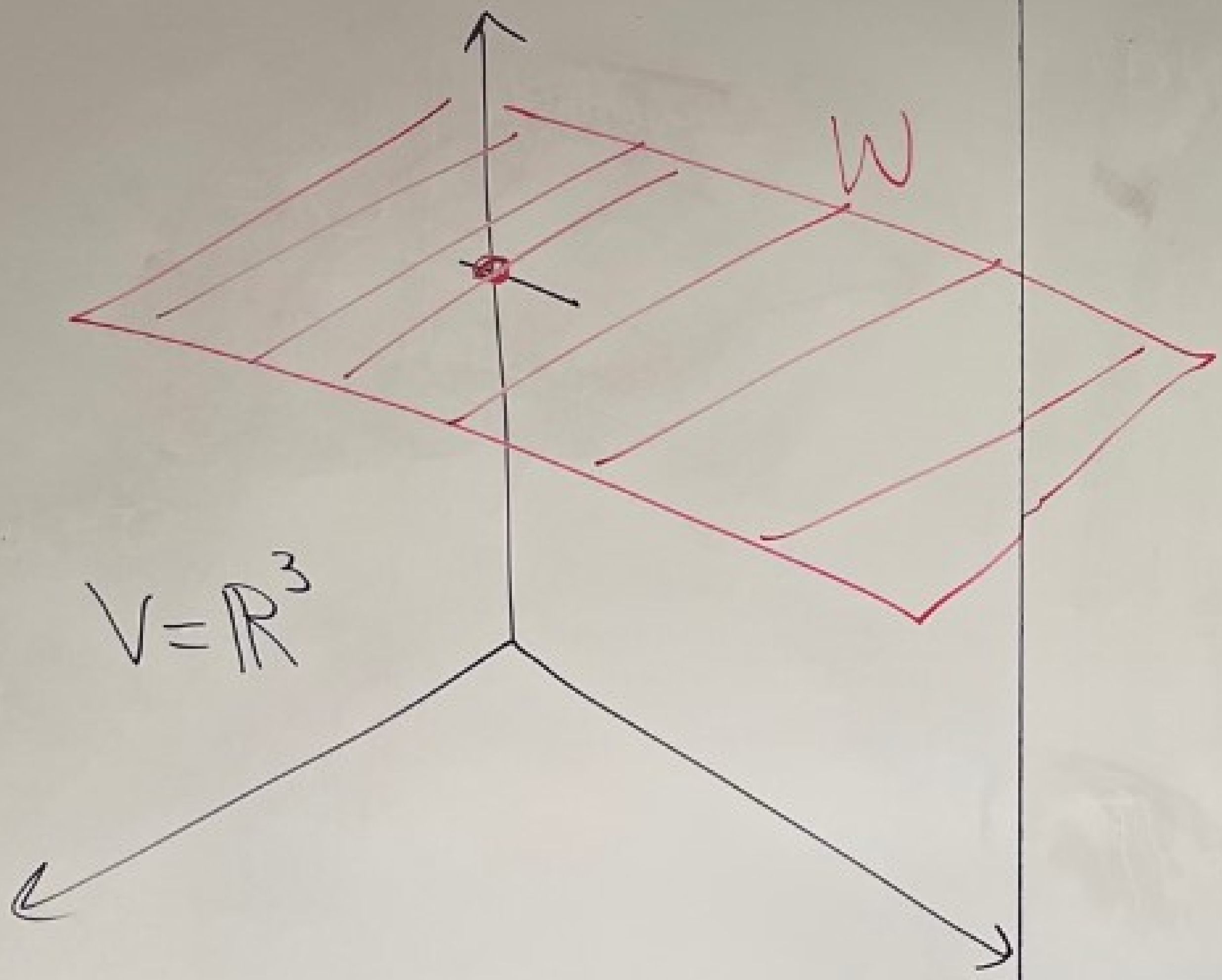
$W = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$

Try to show W is a subspace of $V = \mathbb{R}^3$



Ex: $V=\mathbb{R}^3$, $F=\mathbb{R}$

$W=\{(x,y,1) \mid x,y\in\mathbb{R}\}$



$W=\{(5,2,1),(-1,\pi,1),(1,1,1),(0,0,1),\dots\}$

① $\vec{0}=(0,0,0)\notin W$

not 1

So W is not a subspace.

You could instead show ② or ③ fails.

② $(1,1,1)\in W$

$(0,0,1)\in W$

but $(1,1,1)+(0,0,1)=(1,1,2)\notin W$

not 1

③ $(1,1,1)\in W$

but $2\cdot(1,1,1)=(2,2,2)\notin W$

Note: Let V be a vector space over a field F.

V has at least two subspaces:

these two are not the same unless $V=\{\vec{0}\}$

① $W=\{\vec{0}\}$ ← called the trivial subspace

② $W=V$

$\boxed{\text{HW 2}}$ — Bases of vector spaces

Def: Let V be a vector space over a field F.
Let $v_1, v_2, \ldots, v_n$ be in V.

(1) The span of $v_1, v_2, \ldots, v_n$ is defined to be the set

$$\text{span}(\{v_1, v_2, \ldots, v_n\}) = \{ \underbrace{c_1 v_1 + c_2 v_2 + \cdots + c_n v_n}_{\substack{\text{this is called a} \\ \text{linear combination} \\ \text{of } v_1, v_2, \ldots, v_n}} \mid c_1, c_2, \ldots, c_n \in F \}$$

② If $V = \text{span}(\{v_1, v_2, \dots, v_n\})$ then we say that V is <u>spanned</u> by $v_1, v_2, \dots, v_n$ or that $v_1, v_2, \dots, v_n$ are a <u>spanning set</u> for V.

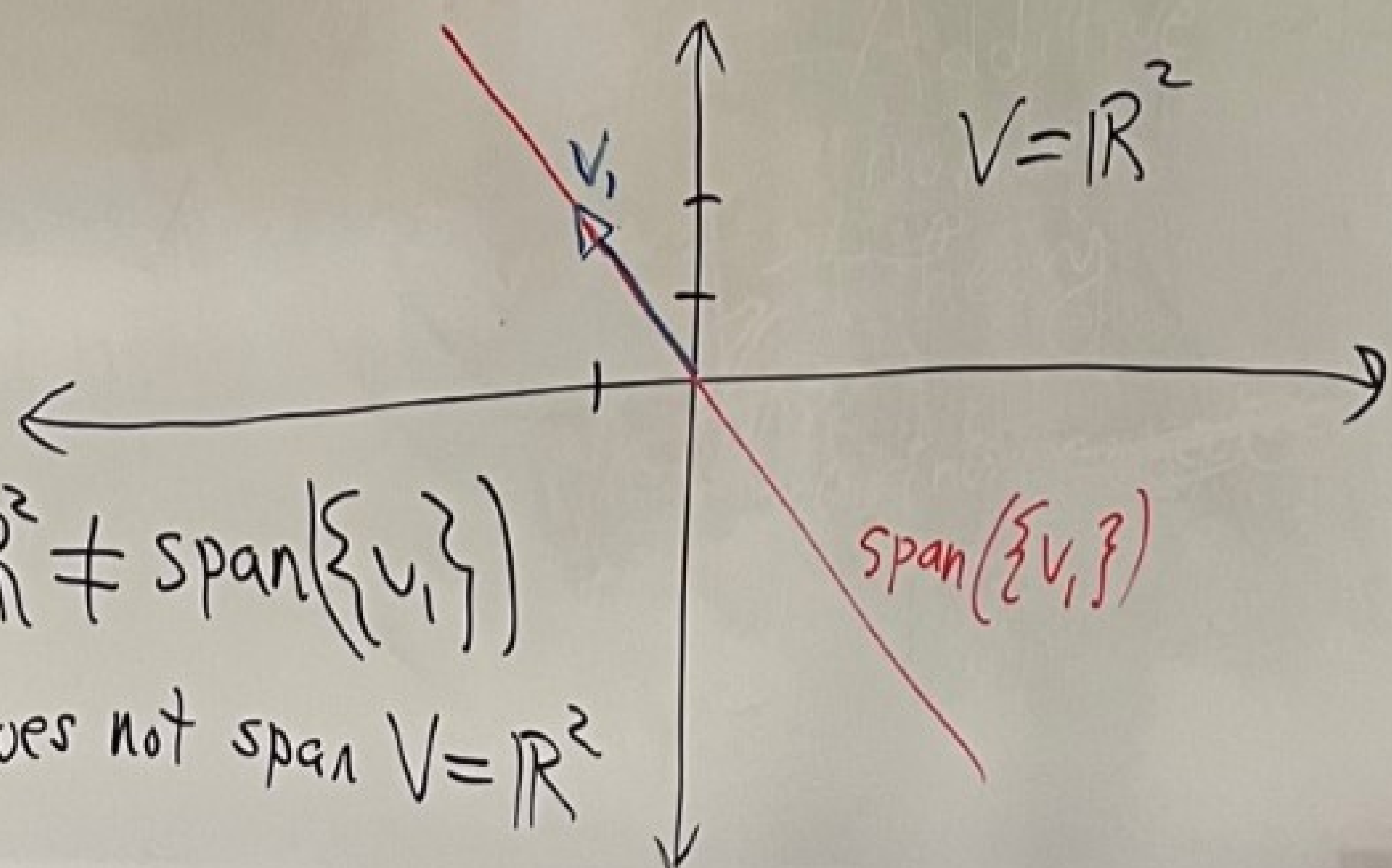
Ex: $V=\mathbb{R}^2$, $F=\mathbb{R}$

Let $v_1 = (-1,2)$.

Then

$$\text{span}(\{v_1\}) = \{ c_1 v_1 \mid c_1 \in \mathbb{R} \}$$

$$= \{ (-c_1, 2c_1) \mid c_1 \in \mathbb{R} \} = \{ \underbrace{(-1,2)}_{c_1=1}, \underbrace{(1,-2)}_{c_1=-1}, \underbrace{(-5,10)}_{c_1=5}, \underbrace{(-\pi, 2\pi)}_{c_1=\pi}, \dots \}$$



Since $\mathbb{R}^2 \neq \text{span}(\{v_1\})$

then v_1 does not span $V=\mathbb{R}^2$

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$.

Let $v_1 = (1,0)$, $v_2 = (0,1)$

Then,

$$\begin{aligned}
\operatorname{span}(\{v_1, v_2\}) &= \{c_1(1,0) + c_2(0,1) \mid c_1, c_2 \in \mathbb{R}\} \\
&= \{(c_1, 0) + (0, c_2) \mid c_1, c_2 \in \mathbb{R}\} \\
&= \{(c_1, c_2) \mid c_1, c_2 \in \mathbb{R}\} = \mathbb{R}^2
\end{aligned}$$

So, $v_1=(1,0)$, $v_2=(0,1)$ span $V=\mathbb{R}^2$.

Every vector in $\mathbb{R}^2$ is a linear combination of v_1, v_2.

For example,

$$\begin{aligned}(1,3) &= (1,0)+(0,3)\\ &= 1\cdot(1,0)+3\cdot(0,1)\\ &= 1\cdot v_1+3\cdot v_2\end{aligned}$$