Ex from last time:

$V = M_{2,2}(\mathbb{R})$, $F = \mathbb{R}$

$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Last time we showed

v_1, v_2, v_3, v_4 span $V = M_{2,2}(\mathbb{R})$.

Are they linearly independent?

Suppose

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \vec{0}$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$ ← $\boxed{F = \mathbb{R}}$

This becomes

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This gives

$$\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So, $\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Thus, $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0$.

The only solution to

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \vec{0}$$

is $\underbrace{(c_1, c_2, c_3, c_4) = (0,0,0,0)}_{\text{another way to write } c_1 = 0,\ c_2 = 0,\ c_3 = 0,\ c_4 = 0.}$

Thus, v_1, v_2, v_3, v_4 are linearly independent.

Since v_1, v_2, v_3, v_4 are lin. ind. and they span $V = M_{2,2}(\mathbb{R})$ over $F = \mathbb{R}$ they are a basis for $V = M_{2,2}(\mathbb{R})$ over $F = \mathbb{R}$.

Notation for next theorem

Consider the system

$$\boxed{\begin{aligned} 10x_1 - 3x_2 + \tfrac{1}{3}x_3 &= 0 \\ 5x_2 - x_3 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}} \qquad (*)$$

Let

$A_1 = (10, -3, \frac{1}{3})$

$A_2 = (0, 5, -1)$

$A_3 = (-1, 1, 0)$

$X = (x_1, x_2, x_3)$

Th

Then, $(*)$ becomes

$$\boxed{\begin{aligned} A_1 \cdot X &= 0 \\ A_2 \cdot X &= 0 \\ A_3 \cdot X &= 0 \end{aligned}}$$

same as $(*)$

If you $\frac{1}{10}R_1 + R_3 \to R_3$ to $(*)$ you get

$$\boxed{\begin{aligned} 10x_1 - 3x_2 + \tfrac{1}{3}x_3 &= 0 \\ 5x_2 - x_3 &= 0 \\ \tfrac{7}{10}x_2 + \tfrac{1}{30}x_3 &= 0 \end{aligned}}$$

This is the same as:

$$\boxed{\begin{aligned} A_1 \cdot X &= 0 \\ A_2 \cdot X &= 0 \\ \left(\tfrac{1}{10}A_1 + A_3\right) \cdot X &= 0 \end{aligned}}$$

$$\leftarrow \left(1-1, \tfrac{-3}{10}+1, \tfrac{1}{30}+0\right) \cdot (x_1, x_2, x_3) = 0$$

$$\left(0, \tfrac{7}{10}, \tfrac{1}{30}\right) \cdot (x_1, x_2, x_3) = 0$$

$$\tfrac{7}{10}x_2 + \tfrac{1}{30}x_3 = 0$$

Theorem: Let

$$\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}} \quad (*)$$

be a system of m linear equations in n unknowns $[x_1, x_2, \ldots, x_n]$ where $a_{ij} \in F$ where F is a field.

If $n > m$, then $(*)$ has a non-trivial solution.

That is, if $n > m$ then there exists a solution $(x_1, x_2, \ldots, x_n) \in F^n$ to $(*)$ where $(x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)$

that is the x_i are not <u>all</u> zero.

0)

Proof: [I think this is from Lang's linear algebra book]

We induct on m (the # of equations).

base case: Suppose $m=1$.

We assume $n > m = 1$.

So, $\underbrace{n}_{\text{\# of variables}} \geq 2$.

So, (*) becomes

$$\boxed{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0} \quad (*)$$

If $a_{11} = a_{12} = \cdots = a_{1n} = 0$ then a non-trivial sol is $x_1 = x_2 = \cdots = x_n = 1$.

Suppose now at least one of the $a_{ij} \neq 0$.

Without loss of generality assume $a_{11} \neq 0$. ← WLOG

means: if it was $a_{12} \neq 0$ or $a_{13} \neq 0$ the same proof will work.

Then (*) becomes

$$\boxed{x_1 = -a_{11}^{-1}(a_{12}x_2 + \cdots + a_{1n}x_n)} \quad (*)$$

Set $x_2 = x_3 = \cdots = x_n = 1$

and $x_1 = -a_{11}^{-1}(a_{12}(1) + \cdots + a_{1n}(1))$

This solves (*) by a non-trivial solution.

We definitely needed $n \geq 2$ to do this.

Thus, the base case is done.

Induction hypothesis

Now assume the theorem is true for any linear system of $m-1$ equations with more than $m-1$ unknowns

Suppose we have a system (*) with m linear equations and $n > \underbrace{m > 1}$ unknowns.

we don't do $m=1$ because we did that in the base case

If all the $a_{ij} = 0$
then $x_1 = x_2 = \cdots = x_n = 1$
This will give a non-trivial solution.

So we can assume there is some $a_{ij} \neq 0$.

By renumbering the equations and variables we get an equivalent system with $a_{11} \neq 0$.

$$\boxed{\begin{aligned} 0x_1 + 5x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}}$$

equivalent to

$$\boxed{\begin{aligned} x_1 - x_2 &= 0 \\ 0x_1 + 5x_2 &= 0 \end{aligned}}$$

$$\boxed{\begin{aligned} 0x_1 + 5x_2 &= 0 \\ 0x_1 + x_2 &= 0 \end{aligned}}$$

equivalent to

$$\boxed{\begin{aligned} 5x_1 + 0x_2 &= 0 \\ x_1 + 0x_2 &= 0 \end{aligned}}$$

solve then flip $x_1 \leftrightarrow x_2$ to solve

Set

$a_{11} \neq 0$

$$A_1 = (a_{11}, a_{12}, \ldots, a_{1n})$$
$$A_2 = (a_{21}, a_{22}, \ldots, a_{2n})$$
$$\vdots \qquad\qquad \vdots$$
$$A_m = (a_{m1}, a_{m2}, \ldots, a_{mn})$$
$$X = (x_1, x_2, \ldots, x_n)$$

Then (*) becomes

$$\boxed{\begin{aligned} A_1 \cdot X &= 0 \\ A_2 \cdot X &= 0 \\ \vdots \quad & \quad \vdots \\ A_m \cdot X &= 0 \end{aligned}} \qquad (**)$$

By subtracting a multiple of the first row and adding it to the rows below it we can eliminate the x_1 in rows 2 thru m.

Doing this to (**) gives:

$$\boxed{\begin{aligned} A_1 \cdot X &= 0 \\ (A_2 - a_{21} a_{11}^{-1} A_1) \cdot X &= 0 \\ &\vdots \\ (A_m - a_{m1} a_{11}^{-1} A_1) \cdot X &= 0 \end{aligned}}$$

no x_1 in these rows

The last equations

$$\boxed{\begin{array}{c} (A_2 - a_{21}a_{11}^{-1}A_1)\cdot X = 0 \\ \vdots \qquad\qquad \vdots \\ (A_m - a_{m1}a_{11}^{-1}A_1)\cdot X = 0 \end{array}} \qquad (***)$$

are a system of $m-1$ equations with $\underbrace{n-1 > m-1}_{\text{Since } n>m}$ unknowns.

By the induction hypothesis we can find a non-trivial solution

$$(x_2, x_3, \ldots, x_n) \neq (0, 0, \ldots, 0)$$

to $(***)$.

Now using this solution $(x_2, \dots, x_n)$ from (***) we can also solve $A_1 \cdot X = 0$ by setting

$$x_1 = -a_{11}^{-1}(a_{12}x_2 + \dots + a_{1n}x_n).$$

Now set our solution

$$X = (x_1, x_2, \dots, x_n)$$

from the above.

This solves $A_1 \cdot X = 0$.

Why does it solve $A_i \cdot X = 0$ when $i \geq 2$?

If $i \geq 2$, then

$$A_i \cdot X \overset{(***)}{=} a_{i1} a_{11} \underbrace{A_1 \cdot X}_{0} = a_{i1} a_{11} [0] = 0.$$

Thus, we have solved

$$\boxed{\begin{array}{l} A_1 \cdot X = 0 \\ A_2 \cdot X = 0 \\ \vdots \\ A_m \cdot X = 0 \end{array}}$$

with a non-trivial solution. By induction we are done. $\blacksquare$