Study guide for test 1 on website. — 10/5

Theorem: Let V be a vector space over a field F.
Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a subset of V.
Then: β is a basis for V if and only if every element $x \in V$ can be written uniquely in the form

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

where $c_1, c_2, \ldots, c_n \in F$. [ie the $c_1, c_2, \ldots, c_n$ are unique]

Proof:

($\Rightarrow$) Let β be a basis for V and $x \in V$.

Since β is a basis, β spans V.

Thus, x is in the span of β.

So,

$$\boxed{x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n} \quad (*)$$

where $c_1, c_2, \ldots, c_n \in F$.

Why are the $c_1, c_2, \ldots, c_n$ unique?

Suppose there exist $d_1, d_2, \ldots, d_n \in F$ where

$$\boxed{x = d_1 v_1 + d_2 v_2 + \cdots + d_n v_n.} \quad (**)$$

$\in V.$

We need to show that

$c_1 = d_1, c_2 = d_2, \ldots, c_n = d_n$

Computing $(*) - (**)$ gives

$$\vec{0} = \underbrace{(c_1 - d_1)}v_1 + \underbrace{(c_2 - d_2)}v_2 + \cdots + \underbrace{(c_n - d_n)}v_n$$

Since $v_1, v_2, \ldots, v_n$ are linearly independent [since β is a basis]

we know $c_1 - d_1 = 0, c_2 - d_2 = 0, \ldots, c_n - d_n = 0.$

So, $c_1 = d_1, c_2 = d_2, \ldots, c_n = d_n.$

Thus, $x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ where $c_1, c_2, \ldots, c_n$ are unique.

($\Leftarrow$) Suppose every $x \in V$ can be written uniquely in the form

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

where $c_1, c_2, \ldots, c_n \in F$.

This tells us that every $x \in V$ is in the span of β.

So β spans V.

Note

$$\underbrace{\vec{0}}_{x} = 0 v_1 + 0 v_2 + \cdots + 0 v_n$$

Our assumption says that the above is the unique way to solve

$$\underbrace{\vec{0}}_{x} = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

So, $\beta = \{v_1, v_2, \ldots, v_n\}$ is a linearly independent set. So β is a basis for V. $\blacksquare$

Theorem: Let V be a vector space over a field F

Let $v_1, v_2, \ldots, v_m \in V$.

Suppose $v_1, v_2, \ldots, v_m$ span V.

Let $w_1, w_2, \ldots, w_n \in V$.

If $n > m$, then $w_1, w_2, \ldots, w_n$ are linearly dependent.

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proof: Since $v_1, v_2, \ldots, v_m$ span V we can write eld a F

$$
\begin{aligned}
w_1 &= a_{11} v_1 + a_{21} v_2 + \cdots + a_{m1} v_m \\
w_2 &= a_{12} v_1 + a_{22} v_2 + \cdots + a_{m2} v_m \\
&\ \ \vdots \qquad\quad \vdots \qquad\quad\ \ \vdots \qquad\qquad\quad \vdots \\
w_n &= a_{1n} v_1 + a_{2n} v_2 + \cdots + a_{mn} v_m
\end{aligned}
$$

where $a_{ij} \in F$.

For any $c_1, c_2, \dots, c_n \in F$ we have that

$$\begin{aligned}
c_1 w_1 + c_2 w_2 + \dots + c_n w_n &= c_1(a_{11} v_1 + a_{21} v_2 + \dots + a_{m1} v_m) \\
&\quad + c_2(a_{12} v_1 + a_{22} v_2 + \dots + a_{m2} v_m) \\
&\quad \vdots \\
&\quad + c_n(a_{1n} v_1 + a_{2n} v_2 + \dots + a_{mn} v_m) \\
&= (c_1 a_{11} + c_2 a_{12} + \dots + c_n a_{1n}) v_1 \\
&\quad + (c_1 a_{21} + c_2 a_{22} + \dots + c_n a_{2n}) v_2 \\
&\quad \vdots \\
&\quad + (c_1 a_{m1} + c_2 a_{m2} + \dots + c_n a_{mn}) v_m
\end{aligned}$$

From the theorem from last week since $n > m$ we know that

$$\boxed{\begin{aligned} c_1 a_{11} + c_2 a_{12} + \dots + c_n a_{1n} &= 0 \\ c_1 a_{21} + c_2 a_{22} + \dots + c_n a_{2n} &= 0 \\ &\vdots \\ c_1 a_{m1} + c_2 a_{m2} + \dots + c_n a_{mn} &= 0 \end{aligned}} \qquad (*)$$

has a non-trivial solution,

Call it $\hat{c}_1, \hat{c}_2, \dots, \hat{c}_n$.

That is, $(\hat{c}_1, \hat{c}_2, \dots, \hat{c}_n) \neq (0, 0, \dots, 0)$

and $\hat{c}_1, \hat{c}_2, \dots, \hat{c}_n$ solve $(*)$.

And,

$$\hat{c}_1 w_1 + \hat{c}_2 w_2 + \cdots + \hat{c}_n w_n = (\hat{c}_1 a_{11} + \hat{c}_2 a_{12} + \cdots + \hat{c}_n a_{1n}) v_1$$
$$+ (\hat{c}_1 a_{21} + \hat{c}_2 a_{22} + \cdots + \hat{c}_n a_{2n}) v_2$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$+ (\hat{c}_1 a_{m1} + \hat{c}_2 a_{m2} + \cdots + \hat{c}_n a_{mn}) v_m$$
$$= 0 v_1 + 0 v_2 + \cdots + 0 v_m = \vec{0}$$

Thus, $\hat{c}_1 w_1 + \hat{c}_2 w_2 + \cdots + \hat{c}_n w_n = \vec{0}$ shows $w_1, w_2, \ldots, w_n$ are lin. dep. since $(\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n) \neq (0, 0, \ldots, 0)$.

$\blacksquare$

Corollary: Let V be a vector space over a field F.

Suppose $\beta_1 = \{v_1, v_2, \dots, v_a\}$ be a basis for V over F,
and $\beta_2 = \{w_1, w_2, \dots, w_b\}$ be a basis for V over F.

Then, $a = b$.

[That is, any two bases for V have the same # of elements]

proof: Since β_1 is a basis for V, β_1 spans V.

If $b > a$, then by the previous theorem

β_2 would be linearly dependent

But β_2 is a basis so this isn't true.

So, $b \leq a$.

Now let's apply the same reasoning but
switch the roles of β_1 and β_2

Since β_2 is a basis for V,
β_2 spans V.

If $a > b$, then by the
previous theorem
β_1 would be linearly dependent.

This can't happen since β_1 is a basis.

Thus, $a \leq b$.

Since $b \leq a$ and $a \leq b$ this shows that $a = b$

$\blacksquare$

The previous Corollary makes the following definition well-defined.

Def: Let V be a vector space over a field F.
We say that V is finite dimensional if V has a basis with a finite number of elements in the basis.
If V is finite dimensional and $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V with n elements, then we say that the dimension of V is n and we write

$$\dim(V) = n.$$

[You can also write $\dim_F(V) = n$ since this def depends on F.]

(def continued...)

A special case is the vector space $V = \{\vec{0}\}$ which has no basis.

We define this vector space to have dimension 0.

$\dots,0)$

$,\dots,0)$

Ex: Let F be a field.

Let $V = F^n$.

V is a vector space over F of dimension n.

proof: Let

$$v_1 = (1,0,0,\dots,0)$$
$$v_2 = (0,1,0,\dots,0)$$
$$\vdots \qquad \vdots$$
$$v_n = (0,0,\dots,1)$$

Called standard basis

Let's show $v_1,\dots,v_n$ is a basis for $V = F^n$.

Let $x \in F^n$.

Then $x = (x_1, x_2, \dots, x_n)$ where $x_i \in F$.

So,

$$x = (x_1, x_2, \dots, x_n) = (x_1, 0, 0, \dots, 0) + (0, x_2, 0, \dots, 0) + \dots + (0, 0, \dots, x_n)$$
$$= x_1(1, 0, \dots, 0) + x_2(0, 1, 0, \dots, 0) + \dots + x_n(0, 0, \dots, 1)$$
$$= x_1 v_1 + x_2 v_2 + \dots + x_n v_n$$

So, $v_1, v_2, \dots, v_n$ span V.

Suppose

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \vec{0}.$$

Then

$$c_1(1,0,\ldots,0) + c_2(0,1,0,\ldots,0) + c_n(0,0,\ldots,1) = (0,0,\ldots,0)$$

So,

$$(c_1,0,\ldots,0) + (0,c_2,\ldots,0) + \cdots + (0,0,\ldots,c_n) = (0,0,\ldots,0)$$

So, $(c_1,c_2,\ldots,c_n) = (0,0,\ldots,0)$.

So, $c_1 = c_2 = \cdots = c_n = 0$

Thus, $v_1, v_2, \ldots, v_n$ are lin. ind. $\blacksquare$