

Study guide for test 1 on website. — 10/5

Theorem: Let V be a vector space over a field F .
Let $\beta = \{v_1, v_2, \dots, v_n\}$ be a subset of V .

Then: β is a basis for V if and only if
every element $x \in V$ can be written
uniquely in the form

$$x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

where $c_1, c_2, \dots, c_n \in F$. [ie the c_1, c_2, \dots, c_n are unique]

Proof:

(\Rightarrow) Let β be a basis for V and $x \in V$.

Since β is a basis, β spans V .

Thus, x is in the span of β .

So,

$$x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n \quad (*)$$

where $c_1, c_2, \dots, c_n \in F$.

Why are the c_1, c_2, \dots, c_n unique?

Suppose there exist $d_1, d_2, \dots, d_n \in F$ where

$$x = d_1 v_1 + d_2 v_2 + \dots + d_n v_n. \quad (**)$$

We need to show that

$$c_1 = d_1, c_2 = d_2, \dots, c_n = d_n$$

Computing $(*) - (**)$ gives

$$\vec{0} = \underbrace{(c_1 - d_1)}_{\uparrow} v_1 + \underbrace{(c_2 - d_2)}_{\uparrow} v_2 + \dots + \underbrace{(c_n - d_n)}_{\uparrow} v_n$$

Since v_1, v_2, \dots, v_n are linearly independent $\left[\begin{array}{l} \text{since } \mathcal{B} \\ \text{is a basis} \end{array} \right]$

we know $c_1 - d_1 = 0, c_2 - d_2 = 0, \dots, c_n - d_n = 0.$

So, $c_1 = d_1, c_2 = d_2, \dots, c_n = d_n.$

Thus, $x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ where c_1, c_2, \dots, c_n are unique.

(\Leftarrow) Suppose every $x \in V$ can be written uniquely in the form

$$x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

where $c_1, c_2, \dots, c_n \in F$.

This tells us that every $x \in V$ is in the span of β .

So β spans V .

Note

$$\underbrace{\vec{0}}_x = 0v_1 + 0v_2 + \dots + 0v_n$$

Our assumption says that the above is the unique way to solve

$$\underbrace{\vec{0}}_x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

So, $\beta = \{v_1, v_2, \dots, v_n\}$ is a linearly independent set.

So β is a basis for V . \square

Theorem: Let V be a vector space over a field F

Let $v_1, v_2, \dots, v_m \in V$.

Suppose v_1, v_2, \dots, v_m span V .

Let $w_1, w_2, \dots, w_n \in V$.

If $n > m$, then w_1, w_2, \dots, w_n are linearly dependent.

proof: Since v_1, v_2, \dots, v_m

span V we can write

w

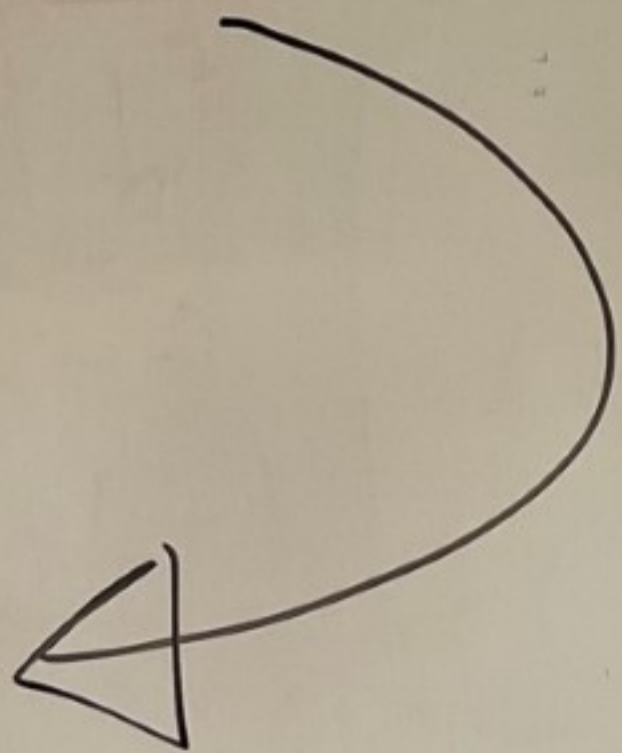
a
old F

$$w_1 = a_{11}v_1 + a_{21}v_2 + \dots + a_{m1}v_m$$

$$w_2 = a_{12}v_1 + a_{22}v_2 + \dots + a_{m2}v_m$$

$$\vdots$$
$$\vdots$$
$$\vdots$$
$$w_n = a_{1n}v_1 + a_{2n}v_2 + \dots + a_{mn}v_m$$

where $a_{ij} \in F$.



For any $c_1, c_2, \dots, c_n \in F$ we have that

$$\begin{aligned} c_1 w_1 + c_2 w_2 + \dots + c_n w_n &= c_1 (a_{11} v_1 + a_{21} v_2 + \dots + a_{m1} v_m) \\ &\quad + c_2 (a_{12} v_1 + a_{22} v_2 + \dots + a_{m2} v_m) \\ &\quad \vdots \\ &\quad + c_n (a_{1n} v_1 + a_{2n} v_2 + \dots + a_{mn} v_m) \end{aligned}$$

$$\begin{aligned} &= (c_1 a_{11} + c_2 a_{12} + \dots + c_n a_{1n}) v_1 \\ &\quad + (c_1 a_{21} + c_2 a_{22} + \dots + c_n a_{2n}) v_2 \\ &\quad \vdots \\ &\quad + (c_1 a_{m1} + c_2 a_{m2} + \dots + c_n a_{mn}) v_m \end{aligned}$$

From the theorem from last week
since $n > m$ we know that

$$c_1 a_{11} + c_2 a_{12} + \dots + c_n a_{1n} = 0$$

$$c_1 a_{21} + c_2 a_{22} + \dots + c_n a_{2n} = 0$$

$$\vdots$$

$$c_1 a_{m1} + c_2 a_{m2} + \dots + c_n a_{mn} = 0$$

(*)

has a non-trivial solution,

call it $\hat{c}_1, \hat{c}_2, \dots, \hat{c}_n$.

That is, $(\hat{c}_1, \hat{c}_2, \dots, \hat{c}_n) \neq (0, 0, \dots, 0)$

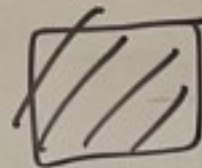
and $\hat{c}_1, \hat{c}_2, \dots, \hat{c}_n$ solve (*).

And,

$$\begin{aligned}\hat{c}_1 w_1 + \hat{c}_2 w_2 + \dots + \hat{c}_n w_n &= (\hat{c}_1 a_{11} + \hat{c}_2 a_{12} + \dots + \hat{c}_n a_{1n}) v_1 \\ &\quad + (\hat{c}_1 a_{21} + \hat{c}_2 a_{22} + \dots + \hat{c}_n a_{2n}) v_2 \\ &\quad \vdots \\ &\quad + (\hat{c}_1 a_{m1} + \hat{c}_2 a_{m2} + \dots + \hat{c}_n a_{mn}) v_m \\ &= 0 v_1 + 0 v_2 + \dots + 0 v_m = \vec{0}\end{aligned}$$

Thus, $\hat{c}_1 w_1 + \hat{c}_2 w_2 + \dots + \hat{c}_n w_n = \vec{0}$ shows

w_1, w_2, \dots, w_n are lin. dep. since $(\hat{c}_1, \hat{c}_2, \dots, \hat{c}_n) \neq (0, 0, \dots, 0)$.



Corollary: Let V be a vector space over a field F .

Suppose $\beta_1 = \{v_1, v_2, \dots, v_a\}$ be a basis for V over F ,

and $\beta_2 = \{w_1, w_2, \dots, w_b\}$ be a basis for V over F .

Then, $a = b$.

[That is, any two bases for V have the same # of elements]

Proof: Since β_1 is a basis for V , β_1 spans V .

If $b > a$, then by the previous theorem

β_2 would be linearly dependent

But β_2 is a basis so this isn't true.

So, $b \leq a$.

Now let's apply the same reasoning but switch the roles of β_1 and β_2

Since β_2 is a basis for V ,
 β_2 spans V .

If $a > b$, then by the
previous theorem

β_1 would be linearly dependent.

This can't happen since
 β_i is a basis.

Thus, $a \leq b$.

Since $b \leq a$ and $a \leq b$
this shows
that $a = b$



The previous Corollary makes the following definition well-defined.

Def: Let V be a vector space over a field F .

We say that V is finite dimensional if V has a basis with a finite number of elements in the basis.

If V is finite dimensional and $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V with n elements, then we say that the dimension of V is n and we write

$$\dim(V) = n.$$

[You can also write $\dim_F(V) = n$ since this def depends on F .]

(def continued...)

A special case is the vector space

$V = \{ \vec{0} \}$ which has no basis.

We define this vector space
to have dimension 0.

Ex: Let F be a field.

$$\text{Let } V = F^n.$$

V is a vector space over F of dimension n .

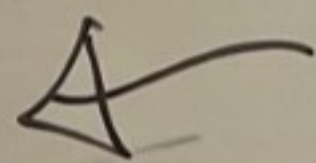
proof: Let

$$v_1 = (1, 0, 0, \dots, 0)$$

$$v_2 = (0, 1, 0, \dots, 0)$$

$$\vdots$$

$$v_n = (0, 0, \dots, 1)$$



Called
standard
basis

Let's show v_1, \dots, v_n is a basis for $V = F^n$.

Let $x \in F^n$.

Then $x = (x_1, x_2, \dots, x_n)$ where $x_i \in F$.

So,

$$\begin{aligned}x &= (x_1, x_2, \dots, x_n) = (x_1, 0, 0, \dots, 0) + (0, x_2, 0, \dots, 0) + \dots + (0, 0, \dots, x_n) \\ &= x_1(1, 0, \dots, 0) + x_2(0, 1, 0, \dots, 0) + \dots + x_n(0, 0, \dots, 1) \\ &= x_1 v_1 + x_2 v_2 + \dots + x_n v_n\end{aligned}$$

So, v_1, v_2, \dots, v_n span V .

Suppose

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = \vec{0}.$$

Then

$$c_1(1, 0, \dots, 0) + c_2(0, 1, 0, \dots, 0) + c_n(0, 0, \dots, 1) = (0, 0, \dots, 0)$$

So,

$$(c_1, 0, \dots, 0) + (0, c_2, \dots, 0) + \dots + (0, 0, \dots, c_n) = (0, 0, \dots, 0)$$

So, $(c_1, c_2, \dots, c_n) = (0, 0, \dots, 0)$.

So, $c_1 = c_2 = \dots = c_n = 0$

Thus, v_1, v_2, \dots, v_n are lin. ind.

