Test 1 - Weds
Mon - can do any HW you want

Last time

Summary: $n \geq 1$

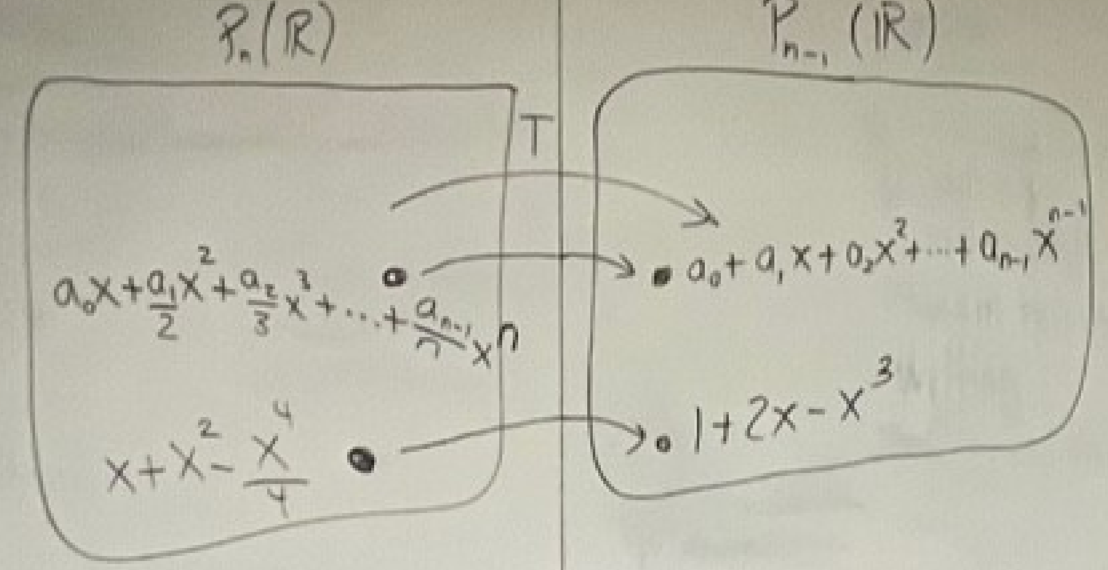
$$T : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$$

$$T(f) = f'$$

$$N(T) = \text{span}(\{1\})$$

$$\dim(N(T)) = 1$$

Let's figure out the range



Claim: T is onto $P_{n-1}(\mathbb{R})$

proof:

Let $a_0+a_1x+a_2x^2+\dots+a_{n-1}x^{n-1}\in P_{n-1}(\mathbb{R})$

Then $a_0x+\frac{a_1}{2}x+\frac{a_2}{3}x^3+\dots+\frac{a_{n-1}}{n}x^n\in P_n(\mathbb{R})$

and

$$T\left(a_0x+\frac{a_1}{2}x+\frac{a_2}{3}x^3+\dots+\frac{a_{n-1}}{n}x^n\right)=a_0+a_1x+a_2x^2+\dots+a_{n-1}x^{n-1}.\ \blacksquare$$

So, $R(T)=P_{n-1}(\mathbb{R})$ and $\operatorname{rank}(T)=\dim(R(T))=(n-1)+1=n$

Note:

$\dim(P_n(\mathbb{R}))=n$

$\dim(\underbrace{P_n(\mathbb{R})}_{\text{domain of }T})=\dim($

$+ \ldots + a_{n-1}x^{n-1}.$ $\blacksquare$

Note:

$$\dim(P_n(\mathbb{R})) = n+1$$

$$\dim(\underbrace{P_n(\mathbb{R})}_{\substack{\text{domain}\\ \text{of } T}}) = \underbrace{\dim(N(T))}_{1} + \underbrace{\dim(R(T))}_{n}$$

$$T) = \dim(R(T)) = (n-1)+1 = n$$

How to make a linear transformation from a matrix.

<u>Def:</u> Let F be a field.

Let A be an $m \times n$ matrix with coefficients from F.

Define

$$L_A : F^n \longrightarrow F^m$$

by $L_A(x) = \underbrace{Ax}_{\text{matrix multiplication}}$ where $x \in F^n$

$$\underset{m \times n}{A} \quad \underset{n \times 1}{x}$$

answer is $m \times 1$
ie in F^m

You could call L_A the <u>left-multiplication by A linear transformation</u>

Claim: L_A is a linear transformation

Proof: Let $x, y \in F^n$ and $\alpha, \beta \in F$.

Then,

$$\begin{aligned}
L_A(\alpha x + \beta y) &= A(\alpha x + \beta y) \\
&= A(\alpha x) + A(\beta y) \\
&= \alpha A x + \beta A y \\
&= \alpha L_A(x) + \beta L_A(y)
\end{aligned}$$

Properties of matrices

$\blacksquare$

"$i=\sqrt{-1}$"

$$\boxed{i^2=-1}$$

Ex: Let $F=\mathbb{C}$.

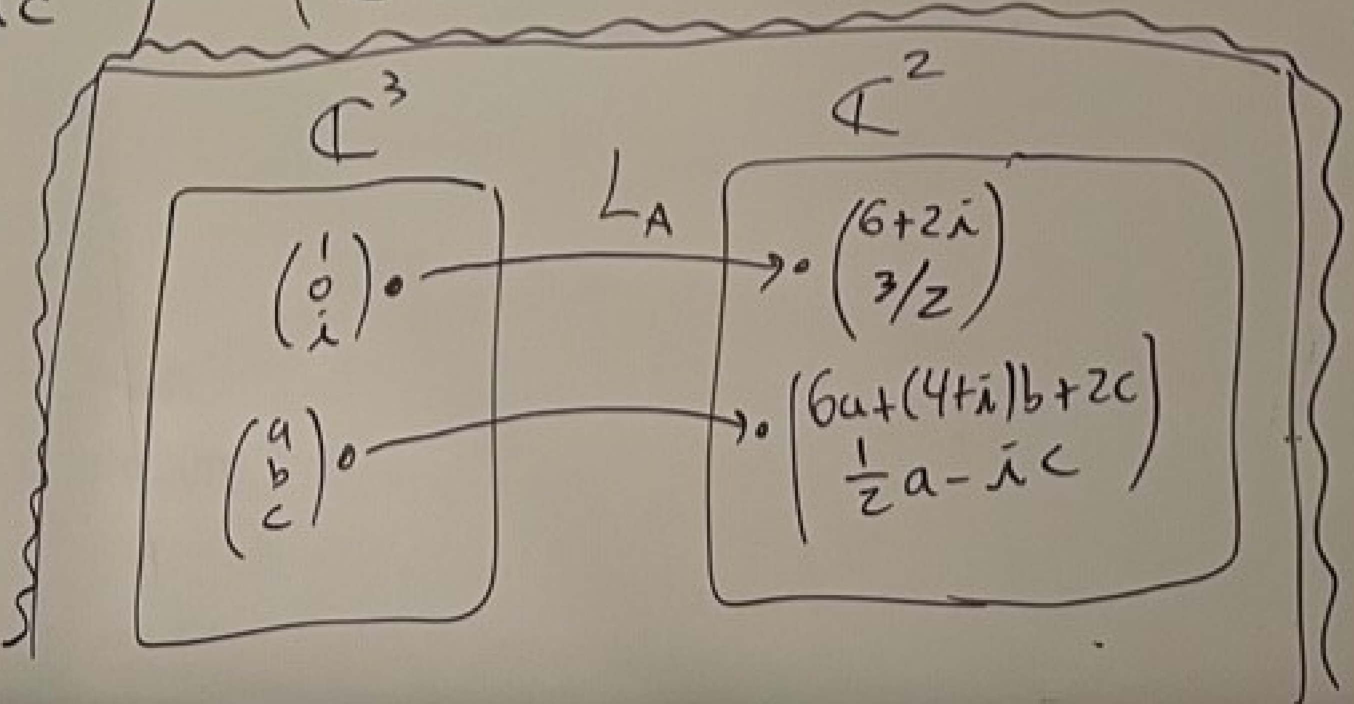
Let
$$A=\begin{pmatrix}6 & 4+i & 2\\ \frac{1}{2} & 0 & -i\end{pmatrix} \text{ is } m\times n = 2\times 3$$

So, $L_A:\mathbb{C}^3\to\mathbb{C}^2$

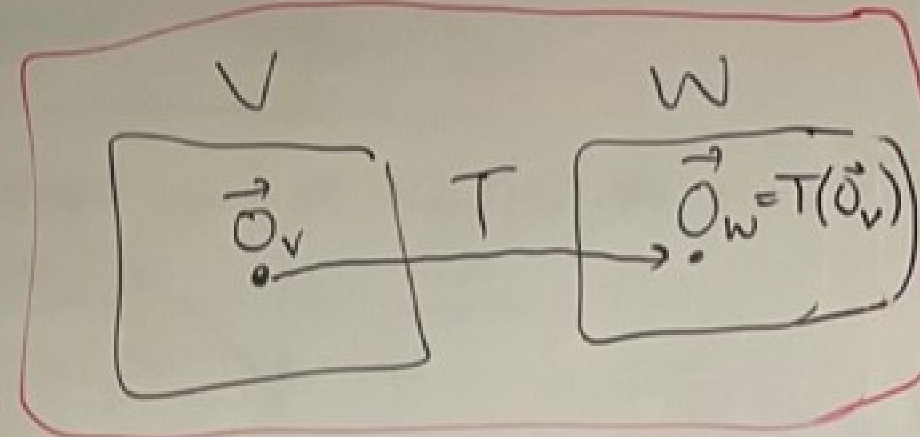
$$L_A\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}6 & 4+i & 2\\ \frac{1}{2} & 0 & -i\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}6a+(4+i)b+2c\\ \frac{1}{2}a+0b-ic\end{pmatrix}=\begin{pmatrix}6a+(4+i)b+2c\\ \frac{1}{2}a-ic\end{pmatrix}$$

For example,

$$L_A\begin{pmatrix}1\\0\\i\end{pmatrix}=\begin{pmatrix}6+(4+i)0+2i\\ \frac{1}{2}(1)-i(i)\end{pmatrix}\underset{\boxed{-i^2=-(-1)=1}}{=}\begin{pmatrix}6+2i\\ 3/2\end{pmatrix}$$



Theorem: Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear transformation. Let $\vec{0}_V$ and $\vec{0}_W$ be the zero vectors of V and W. Then $T(\vec{0}_V) = \vec{0}_W$.



[HW PROBLEM]

proof: We have $T(\vec{0}_V) = T(\vec{0}_V + \vec{0}_V) = T(\vec{0}_V) + T(\vec{0}_V)$

T is linear

Add $-T(\vec{0}_V)$ to both sides to get

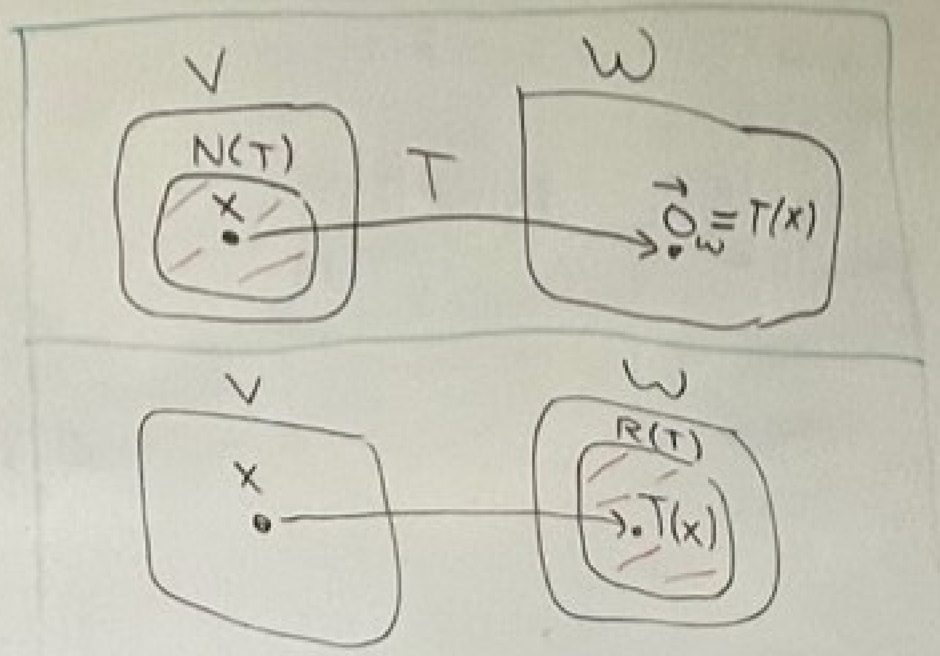
$$\underbrace{-T(\vec{0}_V) + T(\vec{0}_V)}_{\vec{0}_W} = \underbrace{-T(\vec{0}_V) + T(\vec{0}_V)}_{\vec{0}_W} + T(\vec{0}_V)$$

So, $\vec{0}_W = \vec{0}_W + T(\vec{0}_V)$, So, $\vec{0}_W = T(\vec{0}_V)$ $\blacksquare$

Theorem: Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear transformation. Then,

① $N(T)$ is a subspace of V

② $R(T)$ is a subspace of W



Proof of ①: Recall

$$N(T) = \{x \in V \mid T(x) = \vec{0}_W\}$$

Let's show $N(T)$ is a subspace of V.

(i) From the previous theorem $T(\vec{0}_V) = \vec{0}_W$. So, $\vec{0}_V \in N(T)$.

(ii) Let $x, y \in N(T)$

We need to show that $x+y \in N(T)$.

Since $x \in N(T)$ we know $T(x) = \vec{0}_W$

Since $y \in N(T)$ we know $T(y) = \vec{0}_W$

Thus, $T(x+y) \underset{\text{since } T \text{ is linear}}{=} T(x) + T(y) = \vec{0}_W + \vec{0}_W = \vec{0}_W$

So, $x+y \in N(T)$

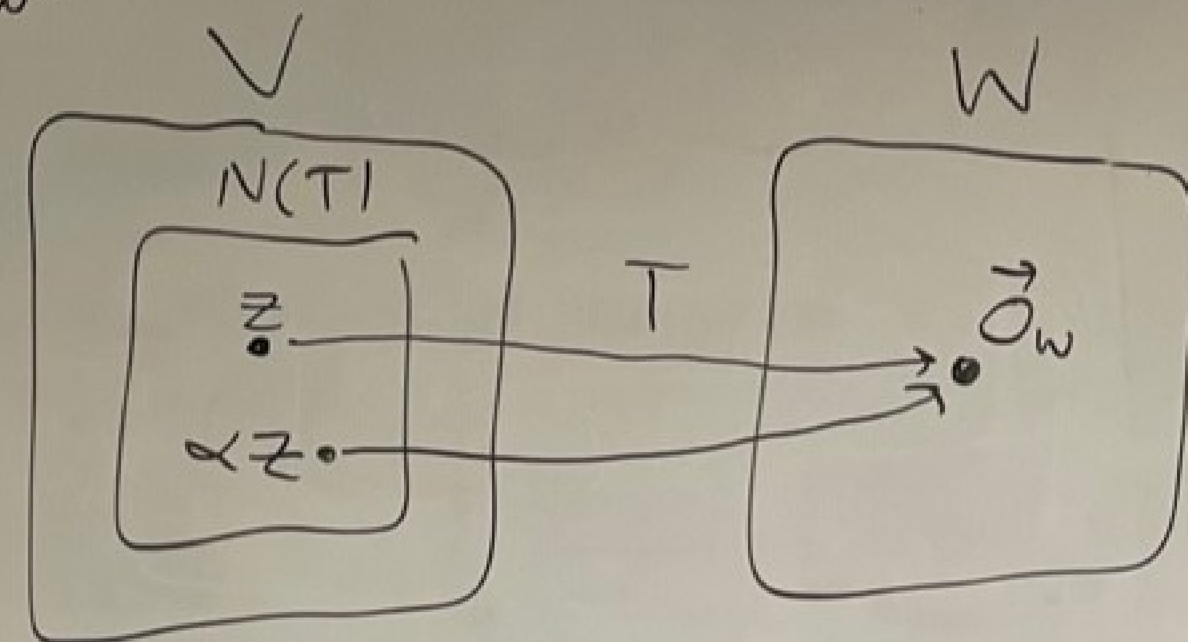
(iii) Let $z \in N(T)$ and $\alpha \in F$.

We must show that $\alpha z \in N(T)$

Since $z \in N(T)$ we know $T(z) = \vec{0}_W$.

Thus,

$$T(\alpha z) \underset{\substack{\uparrow \\ \text{since } T \\ \text{is linear}}}{=} \alpha T(z) = \alpha \vec{0}_W = \vec{0}_W$$



So, $\alpha z \in N(T)$.

By (i), (ii), (iii) $N(T)$ is a subspace of V. $\boxed{1}$

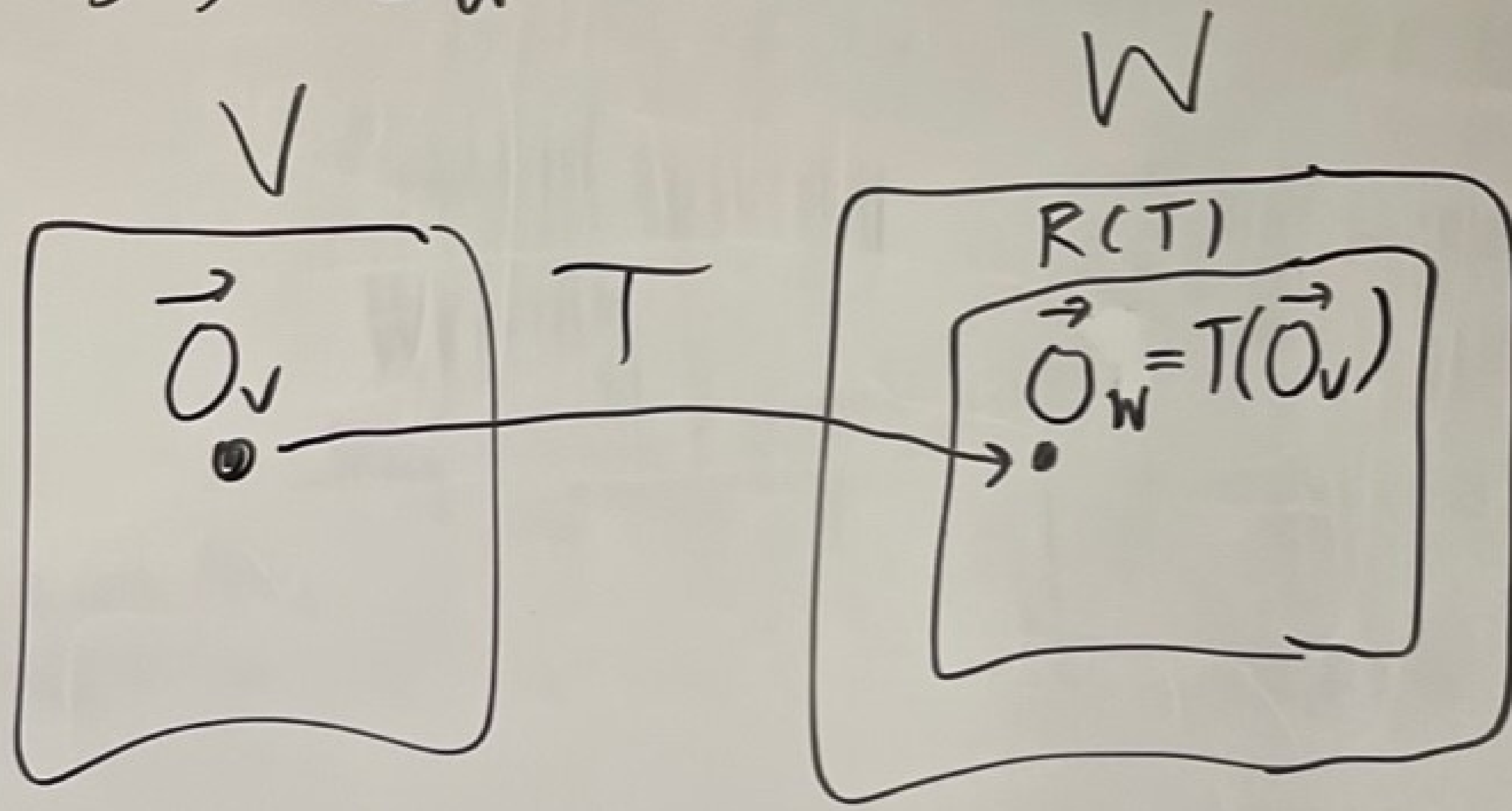
proof of ②: Recall

$$R(T) = \{T(x) \mid x \in V\}$$

Let's show $R(T)$ is a subspace of W.

(i) By the previous theorem, $\vec{0}_W = T(\vec{0}_V)$

So, $\vec{0}_W \in R(T)$.



(ii) Let $a, b \in R(T)$

We need to show that $a+b \in R(T)$

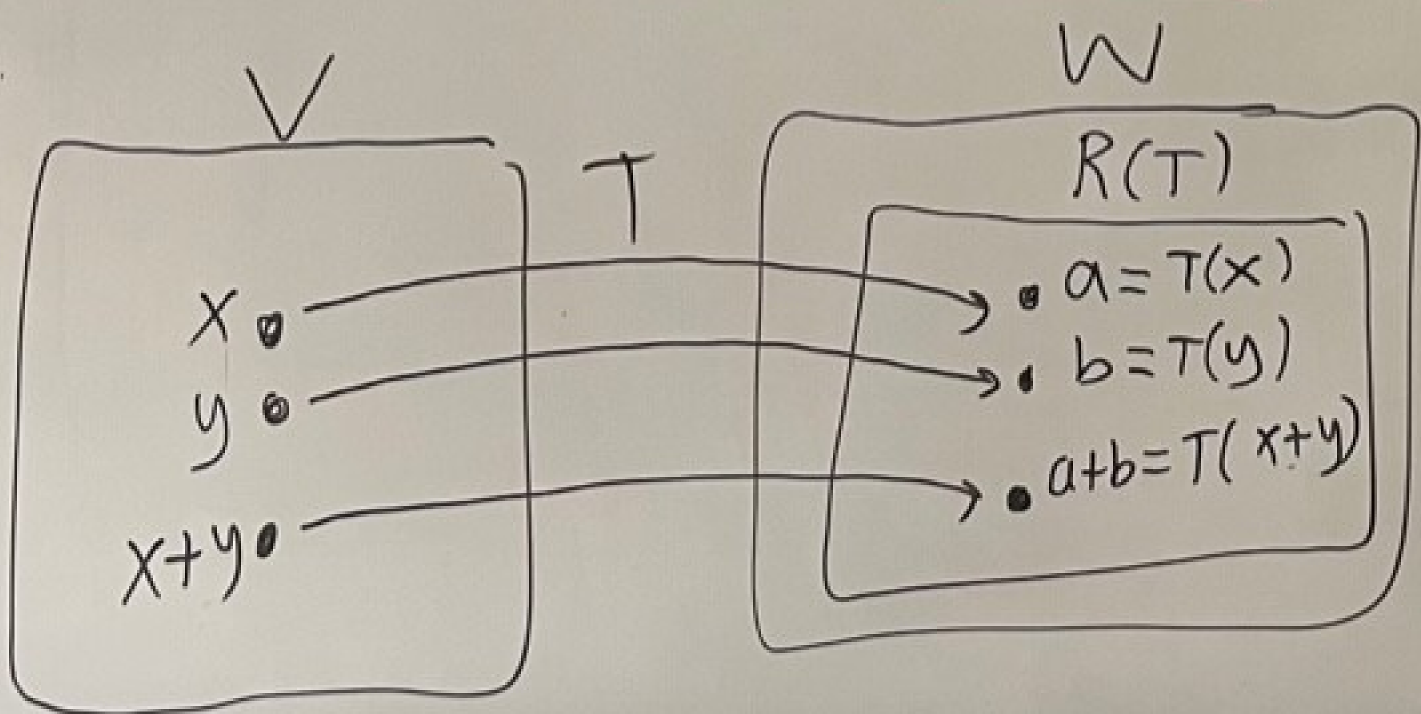
Since $a \in R(T)$ there exists $x \in V$ where $a = T(x)$

Since $b \in R(T)$ there exists $y \in V$ where $b = T(y)$

Since $x, y \in V$ we know $x+y \in V$.

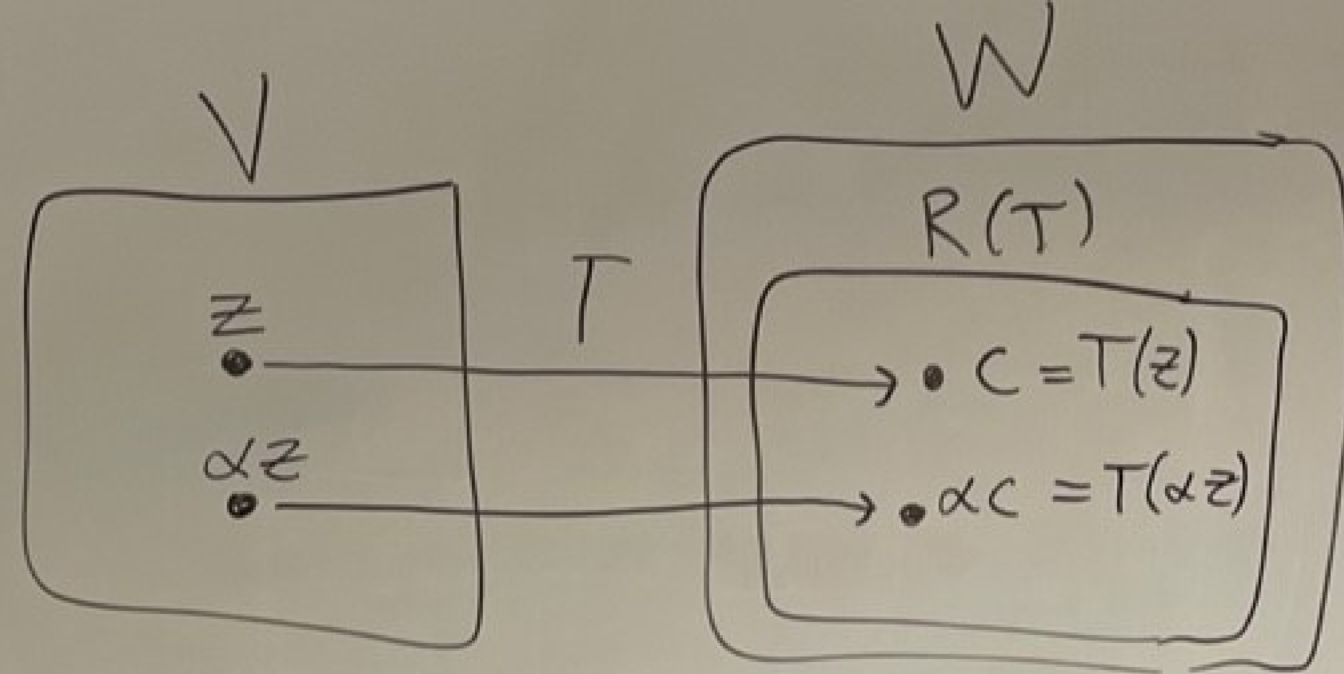
Then, $a+b = T(x)+T(y) \underset{\boxed{T \text{ is linear}}}{=} T(x+y)$.

Since $a+b = T(x+y)$ where $x+y \in V$ we know $a+b \in R(T)$.



(iii) Let $c \in R(T)$ and $\alpha \in F$.

We need to show that $\alpha c \in R(T)$



Since $c \in R(T)$ there exists $z \in V$ where $c = T(z)$

Since $z \in V$ we know $\alpha z \in V$.

And

$$\alpha c = \alpha T(z) \underset{\text{T is linear}}{=} T(\alpha z)$$

Since $\alpha c = T(\alpha z)$ where $\alpha z \in V$ we know $\alpha c \in R(T)$.

By (i), (ii), (iii), $R(T)$ is a subspace of W $\blacksquare$