Last time we showed

$$v_1 = (1,0), \; v_2 = (0,1)$$

span $V = \mathbb{R}^2$ over $F = \mathbb{R}$

because every vector $(a,b) \in \mathbb{R}^2$

can be written as

$$(a,b) = (a,0) + (0,b) = \underbrace{a(1,0) + b(0,1)}$$

linear combo of $v_1 = (1,0)$, $v_2 = (0,1)$
and so is in $\mathrm{span}(\{v_1, v_2\})$

$\underline{\text{Ex}}$: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$.

Let $v_1 = (2,1)$, $v_2 = (-1,1)$.

Do v_1, v_2 span $\mathbb{R}^2$?

You can rephrase this as:

Given any (a,b), can we solve

$$(a,b) = c_1 \underbrace{(2,1)}_{v_1} + c_2 \underbrace{(-1,1)}_{v_2}$$

for c_1, c_2?

This becomes

$$(a,b) = (2c_1, c_1) + (-c_2, c_2)$$

which gives

$$(a,b) = (2c_1 - c_2,\ c_1 + c_2)$$

This gives a linear system

$$\boxed{\begin{aligned} 2c_1 - c_2 &= a \\ c_1 + c_2 &= b \end{aligned}}$$

row operations

1. Multiply a row by a non-zero #
2. Interchange two rows
3. Add a multiple of one row to another row

make this a 1

$$\left(\begin{array}{cc|c} \boxed{2} & -1 & a \\ 1 & 1 & b \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ \boxed{2} & -1 & a \end{array}\right)$$

make this 0

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ 0 & \boxed{-3} & a-2b \end{array}\right)$$

$$\begin{array}{rl} (-2 \quad -2 \mid -2b) & \leftarrow -2R_1 \\ +\,(2 \quad -1 \mid a) & \leftarrow R_2 \\ \hline (0 \quad -3 \mid a-2b) & \leftarrow \text{new } R_2 \end{array}$$

make this a 1

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ 0 & 1 & -\frac{1}{3}a + \frac{2}{3}b \end{array}\right)$$

Translate back to equations:

$$\boxed{c_1} + c_2 = b \quad ①$$
$$\boxed{c_2} = -\tfrac{1}{3}a + \tfrac{2}{3}b \quad ②$$

leading variables: c_1, c_2

free variables: none ← free variables are non-leading variables

Solve for leading variables:

$$c_1 = -c_2 + b \quad ①$$
$$c_2 = -\tfrac{1}{3}a + \tfrac{2}{3}b \quad ②$$

If you had free variables you'd give them a new name like t or s

free variables can take on any value

Now do back substitution.

② $\boxed{c_2 = -\tfrac{1}{3}a + \tfrac{2}{3}b}$

Plug into ① to get

$$\boxed{c_1} = -c_2 + b = -\left(-\tfrac{1}{3}a + \tfrac{2}{3}b\right) + b = \boxed{\tfrac{1}{3}a + \tfrac{1}{3}b}$$

Thus, for any (a,b) we can write

$$\boxed{(a,b) = \underbrace{\left(\tfrac{1}{3}a + \tfrac{1}{3}b\right)}_{c_1}\underbrace{(2,1)}_{v_1} + \underbrace{\left(-\tfrac{1}{3}a + \tfrac{2}{3}b\right)}_{c_2}\underbrace{(-1,1)}_{v_2}}$$

Ex:

$(a,b) = (1,1)$

$(1,1) = \frac{2}{3}(2,1) + \frac{1}{3}(-1,1)$

$(1,1) = \frac{2}{3}v_1 + \frac{1}{3}v_2$

$\boxed{\text{Thus, } v_1, v_2 \text{ span } \mathbb{R}^2}$ ← Answer

Lemma $\boxed{\text{From Hw 1 \#4a}}$

Let V be a vector space over a field F.
Let $\vec{0}$ be the zero vector of V.
Let 0 be the additive identity of F.

Then, $0w = \vec{0}$ for any $w \in V$.

proof: We have that

$$0w \underset{\text{F3}}{=} (0+0)w \underset{\text{V8}}{=} 0w + 0w \qquad (*)$$

By V4 $-0w$ exists and
$-0w + 0w = 0w + (-0w) = \vec{0}$.

So add $-0w$ to both sides of $(*)$ to get

$$\underbrace{-0w + 0w}_{\vec{0}} = \underbrace{-0w + 0w}_{\vec{0}} + 0w$$

So, $\vec{0} = \vec{0} + 0w$.

Thus, $\vec{0} = 0w$. $\blacksquare$

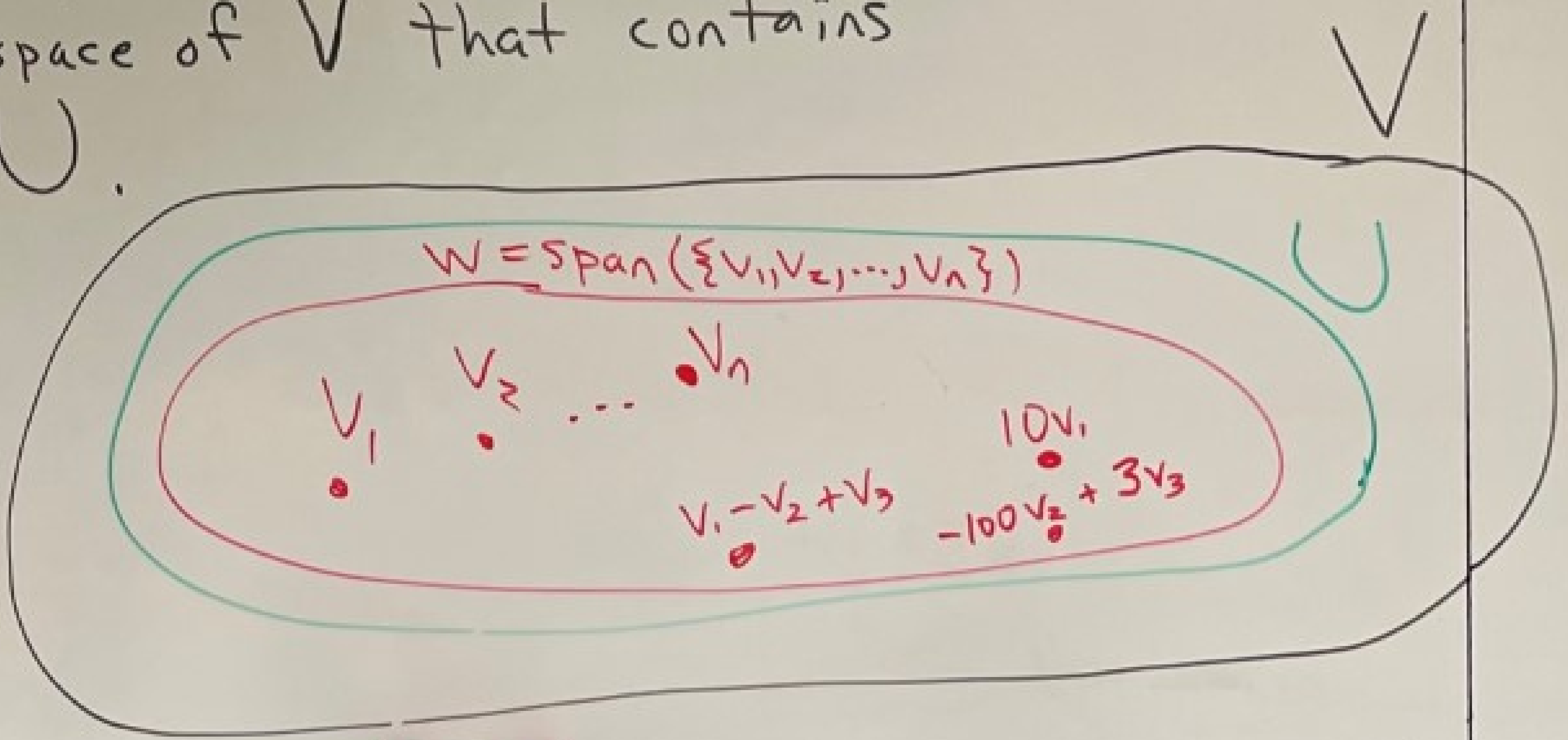
Theorem: Let V be a vector space over a field F.

Let $v_1, v_2, \ldots, v_n$ be in V.

Let $W = \text{span}(\{v_1, v_2, \ldots, v_n\}) = \{ c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \mid c_1, c_2, \ldots, c_n \in F \}$

Then:

1. W is a subspace of V
2. W is the "smallest" subspace of V that contains $v_1, v_2, \ldots, v_n$.
   That is, if U is a subspace of V that contains $v_1, v_2, \ldots, v_n$ then $W \subseteq U$.



proof of ①: Let's show W is a subspace of V

(i) If you set $c_1 = c_2 = \cdots = c_n = 0$ then we get

$$0v_1 + 0v_2 + \cdots + 0v_n \overset{\text{lemma}}{=} \vec{0} + \vec{0} + \cdots + \vec{0}$$
$$= \vec{0}$$

Thus, $\vec{0} = \underbrace{0v_1 + 0v_2 + \cdots + 0v_n}_{\text{in span}(\{v_1, v_2, \ldots, v_n\})} \in W.$

(ii) Let's show W is closed under $+$.

Let $w_1, w_2 \in W$.

Then,

$$w_1 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$
and
$$w_2 = d_1 v_1 + d_2 v_2 + \cdots + d_n v_n$$

where $c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_n \in F$.

Thus,

$$w_1 + w_2 = (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) + (d_1 v_1 + d_2 v_2 + \cdots + d_n v_n)$$
$$= (c_1 + d_1) v_1 + (c_2 + d_2) v_2 + \cdots + (c_n + d_n) v_n$$

linear combo of $v_1, v_2, \ldots, v_n$

So, $w_1 + w_2 \in W$.

(iii) Let's show W is closed under scaling.

Let $w \in W$, and $\alpha \in F$.

Then, $w = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

where $c_1, c_2, \ldots, c_n \in F$.

So,

$$\begin{aligned} \alpha w &= \alpha(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) \\ &= \alpha c_1 v_1 + \alpha c_2 v_2 + \cdots + \alpha c_n v_n \\ &= (\alpha c_1) v_1 + (\alpha c_2) v_2 + \cdots + (\alpha c_n) v_n \end{aligned}$$

linear combo of $v_1, v_2, \ldots, v_n$

Thus, $\alpha w \in W$.

By (i), (ii), (iii), we know W is a subspace of V.

proof of ②: Let U be a subspace of V
where $v_1, v_2, \dots, v_n \in U$.
We need to show that $W \subseteq U$.

Let $w \in W$.
Since $W = \text{span}(\{v_1, v_2, \dots, v_n\})$
we know $w = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$
where $c_1, c_2, \dots, c_n \in F$.

Since $v_1, v_2, \dots, v_n \in U$ and U is a subspace of V we know that
$c_1 v_1, c_2 v_2, \dots, c_n v_n \in U$
[U is closed under scaling since its a subspace]

Since $c_1 v_1, c_2 v_2, \dots, c_n v_n \in U$ and U is a subspace we know
$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \in U$.
[Since U is closed under + because its a subspace.]

Thus, $w = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \in U$.

So, $W \subseteq U$. ▨