① (a)

We have that

$$T(0_V) = T(0_V + 0_V) \overset{\boxed{\text{since } T \text{ is linear}}}{=} T(0_V) + T(0_V)$$

Add $-T(0_V)$ to both sides to get

$$\underbrace{-T(0_V) + T(0_V)}_{0_W} = \underbrace{-T(0_V) + T(0_V)}_{0_W} + T(0_V)$$

So,

$$0_W = T(0_V)$$

---

① (b) This follows from 1(c) with $n=2$

---

① (c)

($\Rightarrow$) Suppose that T is a linear transformation.

We prove this by induction.

If $n=1$, then $T(\alpha_1 x_1) = \alpha_1 T(x_1)$ by the def of linear transformation.

Suppose $k \geq 1$ and

$$T\left(\sum_{i=1}^{k} \alpha_i x_i\right) = \sum_{i=1}^{k} \alpha_i T(x_i) \quad \left[\text{induction hypothesis}\right]$$

Then,

$$T\left(\sum_{i=1}^{k+1}\alpha_i x_i\right) = T\left(\left(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k\right) + \left(\alpha_{k+1} x_{k+1}\right)\right)$$

def of linear trans.

$$= T(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k) + T(\alpha_{k+1} x_{k+1})$$

$$= T(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k) + \alpha_{k+1} T(x_{k+1})$$

by induction hypothesis

$$= \alpha_1 T(x_1) + \alpha_2 T(x_2) + \cdots + \alpha_k T(x_k) + \alpha_{k+1} T(x_{k+1})$$

$$= \sum_{i=1}^{k+1} \alpha_i T(\alpha_i)$$

By induction, the formula is true for all n.

($\Leftarrow$) Suppose $T\left(\sum_{i=1}^{n}\alpha_i x_i\right) = \sum_{i=1}^{n}\alpha_i T(x_i)$

for all $x_1, \ldots, x_n \in V$ and $\alpha_1, \ldots, \alpha_n \in F$

and any $n \geq 1$

Setting $\alpha_1 = \alpha_2 = 1$ and $n=2$ gives

$$T(x_1 + x_2) = T(x_1) + T(x_2)$$

setting $n=1$ gives

$$T(\alpha_1 x_1) = \alpha_1 T(x_1)$$

Thus T is a linear transformation.

② (a)

<u>T is linear</u>

Let $x=(a,b,c)$ and $y=(d,e,f)$ be in $\mathbb{R}^2$ and $\alpha,\beta\in\mathbb{R}$.

Then,

$$T(\alpha x+\beta y)=T(\alpha(a,b,c)+\beta(d,e,f))$$
$$=T(\alpha a+\beta d,\alpha b+\beta e,\alpha c+\beta f)$$
$$=(\alpha a+\beta d-\alpha b-\beta e,\ 2(\alpha c+\beta f))$$
$$=(\alpha a+\beta d-\alpha b-\beta e,\ 2\alpha c+2\beta f)$$
$$=(\alpha a+\beta d-\alpha b-\beta e,$$

and

$$\alpha T(x)+\beta T(y)=\alpha T(a,b,c)+\beta T(d,e,f)$$
$$=\alpha(a-b,2c)+\beta(d-e,2f)$$
$$=(\alpha a-\alpha b,\ 2\alpha c)+(\beta d-\beta e,2\beta f)$$
$$=(\alpha a-\alpha b+\beta d-\beta e,\ 2\alpha c+2\beta f)$$
$$=(\alpha a-\alpha b+\beta d-\beta e,\ 2\alpha c+2\beta f).$$
$$=(\alpha a+\beta d-\alpha b-\beta e,$$

Note that $T(\alpha x+\beta y)=\alpha T(x)+\beta T(y)$

(i) Let's compute $N(T)$

Note that $(a,b,c) \in N(T)$

iff $T(a,b,c) = (0,0)$

iff $(a-b, 2c) = (0,0)$

iff $\boxed{\begin{array}{l} a-b=0 \\ 2c=0 \end{array}}$

So we need to solve this system: $\boxed{\begin{array}{rr} a-b & =0 \\ & 2c=0 \end{array}}$

This is equivalent to $\boxed{\begin{array}{rr} a-b & =0 \\ & c=0 \end{array}}$ ① ②

This is a reduced system. with leading variables a, c and free variable b.

Set $b = t$.

The solutions are ① $c=0$
② $a=b=t$

Thus, $(a,b,c) \in N(T)$
iff $(a,b,c) = (t,t,0) = t(1,1,0)$

A basis for $N(T)$ is $\beta = \{(1,1,0)\}$

(ii) from part i we get that
nullity$(T) = \dim(N(T)) = 1$

(iii) From HW 3 #6(a) we know that T is 1-1 iff $\dim(N(T)) = 0$. Since this isn't the case, T is not 1-1.

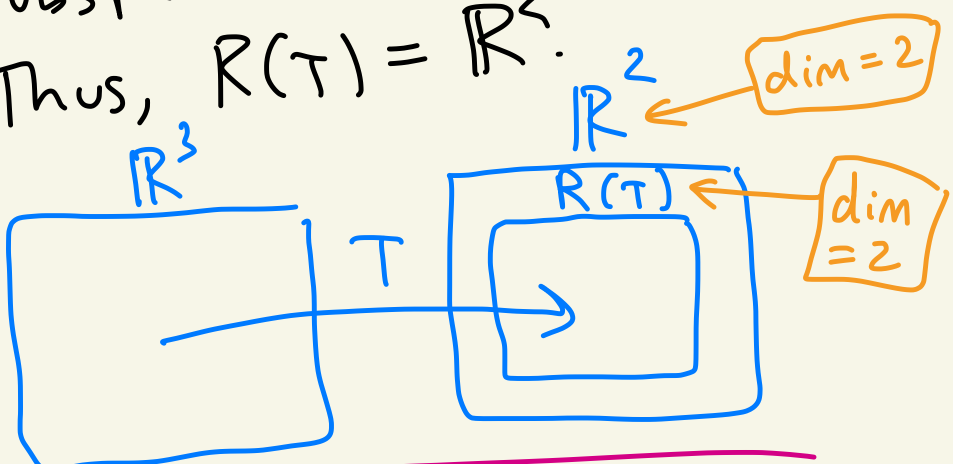
(iv) From the rank-nullity theorem,

$$\dim(\mathbb{R}^3) = \dim(N(T)) + \dim(R(T))$$

So, $3 = 1 + \dim(R(T))$.

Thus, $\dim(R(T)) = 2$.

(v) Since $T: \mathbb{R}^3 \to \mathbb{R}^2$ we see that $R(T)$ is a 2-dimensional subspace of the 2-dimensional vector space $\mathbb{R}^2$. Thus, $R(T) = \mathbb{R}^2$.
So, T is onto.



(vi) As seen in part v, $R(T) = \mathbb{R}^2$

(2) (b)

T is not linear.

For example,

$$T((1,1)+(1,2)) = T(2,3) = (2-3, 3^2) = (-1,9)$$

and

$$T(1,1)+T(1,2) = (1-1, 1^2) + (1-2, 2^2) = (0,1)+(-1,4)$$
$$= (-1,5)$$

Thus,

$$T((1,1)+(1,2)) \neq T(1,1)+T(1,2)$$

You could also do something like this:

$$T(2\cdot(1,1)) = T(2,2) = (2-2, 2^2) = (0,4)$$

and

$$2\cdot T(1,1) = 2\cdot(1-1, 1^2) = 2\cdot(0,1) = (0,2)$$

So, $T(2\cdot(1,1)) \neq 2\cdot T(1,1)$

② (c)

T is linear

Let $x = \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{pmatrix}$ and $y = \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{pmatrix}$

and $\alpha, \beta \in \mathbb{R}$.

Then,

$$T(\alpha x + \beta y) = T\left(\begin{pmatrix} \alpha a_1 & \alpha b_1 & \alpha c_1 \\ \alpha d_1 & \alpha e_1 & \alpha f_1 \end{pmatrix} + \begin{pmatrix} \beta a_2 & \beta b_2 & \beta c_2 \\ \beta d_2 & \beta e_2 & \beta f_2 \end{pmatrix} \right)$$

$$= T\begin{pmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 & \alpha c_1 + \beta c_2 \\ \alpha d_1 + \beta d_2 & \alpha e_1 + \beta e_2 & \alpha f_1 + \beta f_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2(\alpha a_1 + \beta a_2) - (\alpha b_1 + \beta b_2) & (\alpha c_1 + \beta c_2) + 2(\alpha d_1 + \beta d_2) \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\alpha a_1 - \alpha b_1 & \alpha c_1 + 2\alpha d_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2\beta a_2 - \beta b_2 & \beta c_2 + 2\beta d_2 \\ 0 & 0 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} 2a_1 - b_1 & c_1 + 2d_1 \\ 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 2a_2 - b_2 & c_2 + 2d_2 \\ 0 & 0 \end{pmatrix}$$

$$= \alpha T(x) + \beta T(y)$$

(i) $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \in N(T)$

iff $T\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

iff $\begin{pmatrix} 2a-b & c+2d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

iff $\boxed{\begin{aligned} 2a-b &= 0 \\ c+2d &= 0 \end{aligned}}$

iff $\boxed{\begin{aligned} a-\tfrac{1}{2}b &= 0 \quad (1) \\ c+2d &= 0 \quad (2) \end{aligned}}$

e, f arent in system so are free

This is a reduced system with
leading variables: a, c
free variables: b, d, e, f

set $b=t, d=s, e=u, f=v$ and then solve (1), (2).

(2) gives $c=-2d=-2s$

(1) gives $a=\frac{1}{2}b=\frac{1}{2}t$

Thus, $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \in N(T)$

iff $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} \frac{1}{2}t & t & -2s \\ s & u & v \end{pmatrix}$

iff $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = t\begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + s\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$

$+ u\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + v\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

where t, s, u, v can be any real #s.

Let $\beta = \left\{ \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$

Then, $N(T) = \mathrm{span}(\beta)$ from above.

You should also check that β is a linearly independent set.

You do this part

Thus, β is a basis for $N(T)$

(ii) From i, $\dim(N(T)) = 4$

(iii) From problem 6(a), T is 1-1 iff $\dim(N(T)) = 0$.
Since $\dim(N(T)) = 4$ we know that T is not 1-1.

(iv) By the rank-nullity theorem

$$\dim(M_{2,3}(\mathbb{R})) = \dim(N(T)) + \dim(R(T))$$

So, $$6 = 4 + \dim(R(T)).$$

Thus, $\dim(R(T)) = 2$

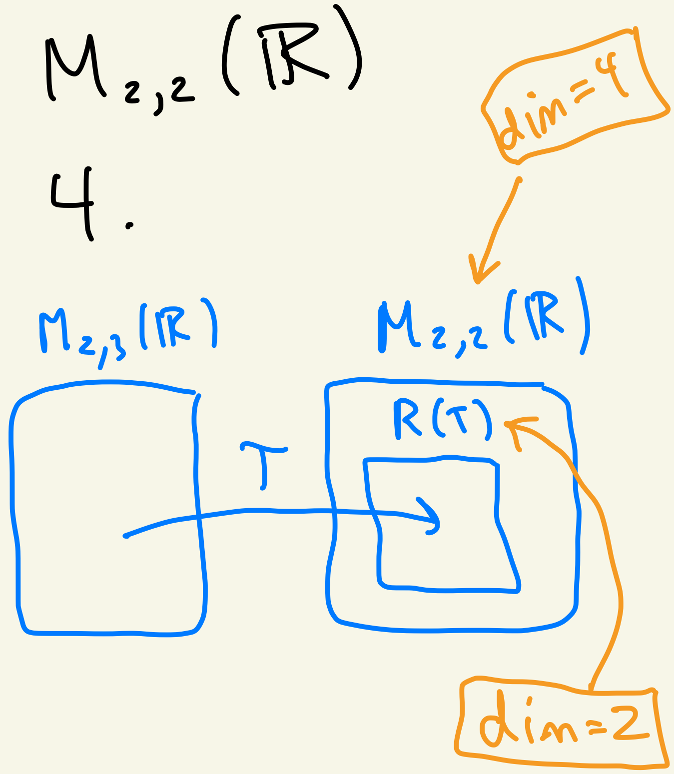
(v) $R(T)$ is dimension 2

$R(T)$ is a subspace of $M_{2,2}(\mathbb{R})$

$M_{2,2}(\mathbb{R})$ has dimension 4.

Thus, $R(T) \neq M_{2,2}(\mathbb{R})$

So, T is not onto.



(vi)

<u>Claim:</u> Let $S = \left\{ \begin{pmatrix} m & n \\ 0 & 0 \end{pmatrix} \,\middle|\, m, n \in \mathbb{R} \right\}$

Then, $R(T) = S$.

<u>pf:</u> First note that

$$T\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 2a-b & c+2d \\ 0 & 0 \end{pmatrix} \in S.$$

So, $R(T) \subseteq S$.

Let $\begin{pmatrix} m & n \\ 0 & 0 \end{pmatrix} \in S$

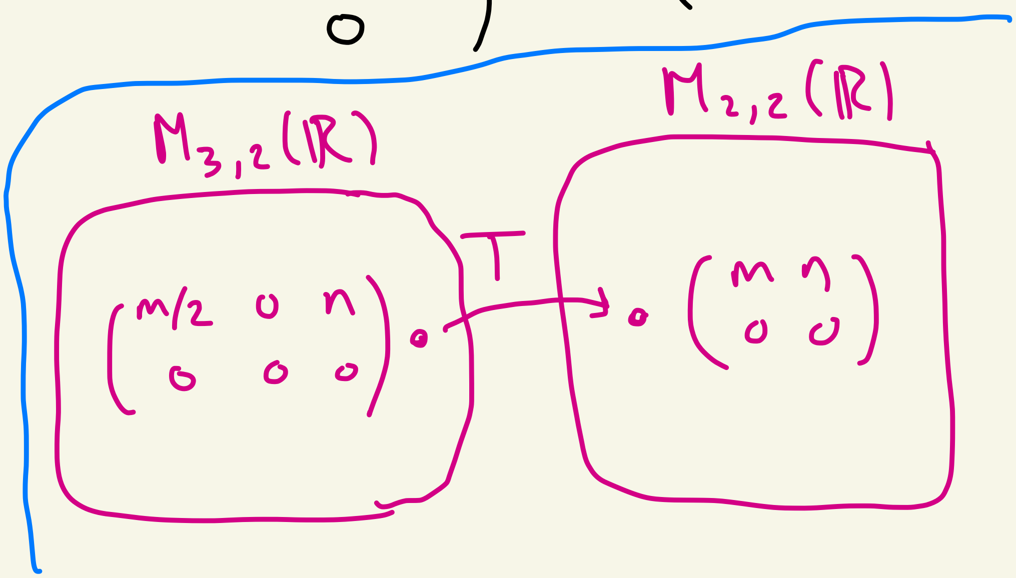
Set $a = \frac{1}{2}m$, $b = 0$, $c = n$, $d = 0$, $e = 0$, $f = 0$.

Then, $T\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = T\begin{pmatrix} m/2 & 0 & n \\ 0 & 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 2(m/2) - 0 & n + 2(0) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} m & n \\ 0 & 0 \end{pmatrix}$$

So, $S \subseteq R(T)$.

$\blacksquare$



② (d)

<u>T is linear</u>

Let $f_1 = a + bx + cx^2$, $f_2 = d + ex + fx^2 \in P_2(\mathbb{R})$
and $\alpha, \beta \in \mathbb{R}$.

Then,

$$T(\alpha f_1 + \alpha f_2) = T(\alpha a + \alpha bx + \alpha cx^2 + \beta d + \beta ex + \beta fx^2)$$

$$= T((\alpha a + \beta d) + (\alpha b + \beta e)x + (\alpha c + \beta f)x^2)$$

$$= (\alpha a + \beta d) + (\alpha b + \beta e)x^3$$

$$= (\alpha a + \alpha bx^3) + (\beta d + \beta ex^3)$$

$$= \alpha(a + bx^3) + \beta(d + ex^3)$$

$$= \alpha T(a + bx + cx^2) + \beta T(d + ex + fx^2)$$

$$= \alpha T(f_1) + \beta T(f_2)$$

$\blacksquare$

(i) $a+bx+cx^2 \in N(T)$

iff $T(a+bx+cx^2) = 0+0x+0x^2+0x^3$

iff $a+bx^3 = 0+0x+0x^2+0x^3$

iff $a=0, b=0$, c is any real number

Thus,

$N(T) = \{cx^2 \mid c \in \mathbb{R}\} = \text{span}(\{x^2\})$

Thus, $\beta = \{x^2\}$ is a basis for $N(T)$.

(ii) By part i, $\dim(N(T)) = 1$

(iii) By HW problem 6a,

T is 1-1 iff $\dim(N(T)) = 0$.

Since this isn't the case

T is not 1-1.

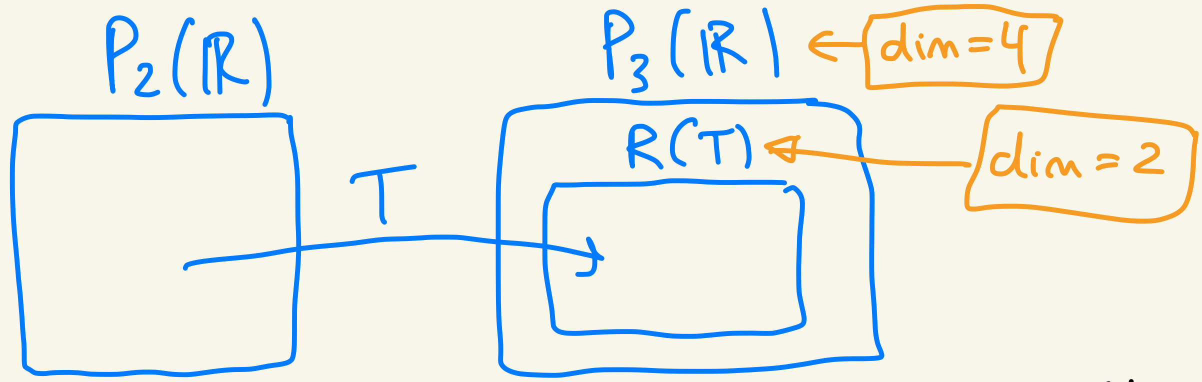
(iv) By the rank-nullity theorem

$$\dim(P_2(\mathbb{R})) = \dim(N(T)) + \dim(R(T))$$

So, $3 = 1 + \dim(R(T))$.

So, $\dim(R(T)) = 2$

(v)



Since $R(T)$ is 2-dimensional and sits inside a 4-dimensional space $P_3(\mathbb{R})$ we have $R(T) \neq P_3(\mathbb{R})$

So, T is not onto.

(vi) $R(T) = \{ a + bx^3 \mid a, b \in \mathbb{R} \}$

$$= \operatorname{span}(\{1, x^3\})$$

So, $\{1, x^3\}$ is a basis for $R(T)$.

(2) (e) T is not linear.

For example,

$$
\begin{aligned}
T(\vec{0}) &= T(0 + 0x + 0x^2) \\
&= (1+0) + (1+0)x + (1+0)x^2 \\
&= 1 + x + x^2 \neq \vec{0}.
\end{aligned}
$$

Recall a linear transformation sends the zero vector to the zero vector by problem 1(a).

(3)

Let $f, g \in C(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$.

Then,

$$T(\alpha f + \beta g) = \int_a^b \left(\alpha f(t) + \beta g(t)\right) dt$$

$$= \alpha \int_a^b f(t)\,dt + \beta \int_a^b g(t)\,dt$$

$$= \alpha T(f) + \beta T(g).$$

So, T is a linear transformation.

④ (a)

$$L_A(x) = \begin{pmatrix} 1 & \pi \\ \frac{1}{2} & -10 \end{pmatrix} \begin{pmatrix} 17 \\ -5 \end{pmatrix} = \begin{pmatrix} 17-5\pi \\ \frac{17}{2}+50 \end{pmatrix} = \begin{pmatrix} 17-5\pi \\ \frac{117}{2} \end{pmatrix}$$

(b)

$$L_A(x) = \begin{pmatrix} -i & 1 & 0 \\ 1+i & 0 & -1 \end{pmatrix} \begin{pmatrix} -2i \\ 4 \\ 1.57 \end{pmatrix}$$

$$= \begin{pmatrix} -i(-2i)+4+0 \\ (1+i)(-2i)+0-1.57 \end{pmatrix}$$

$i^2=-1$

$$= \begin{pmatrix} 2 \\ -2i+2-1.57 \end{pmatrix}$$

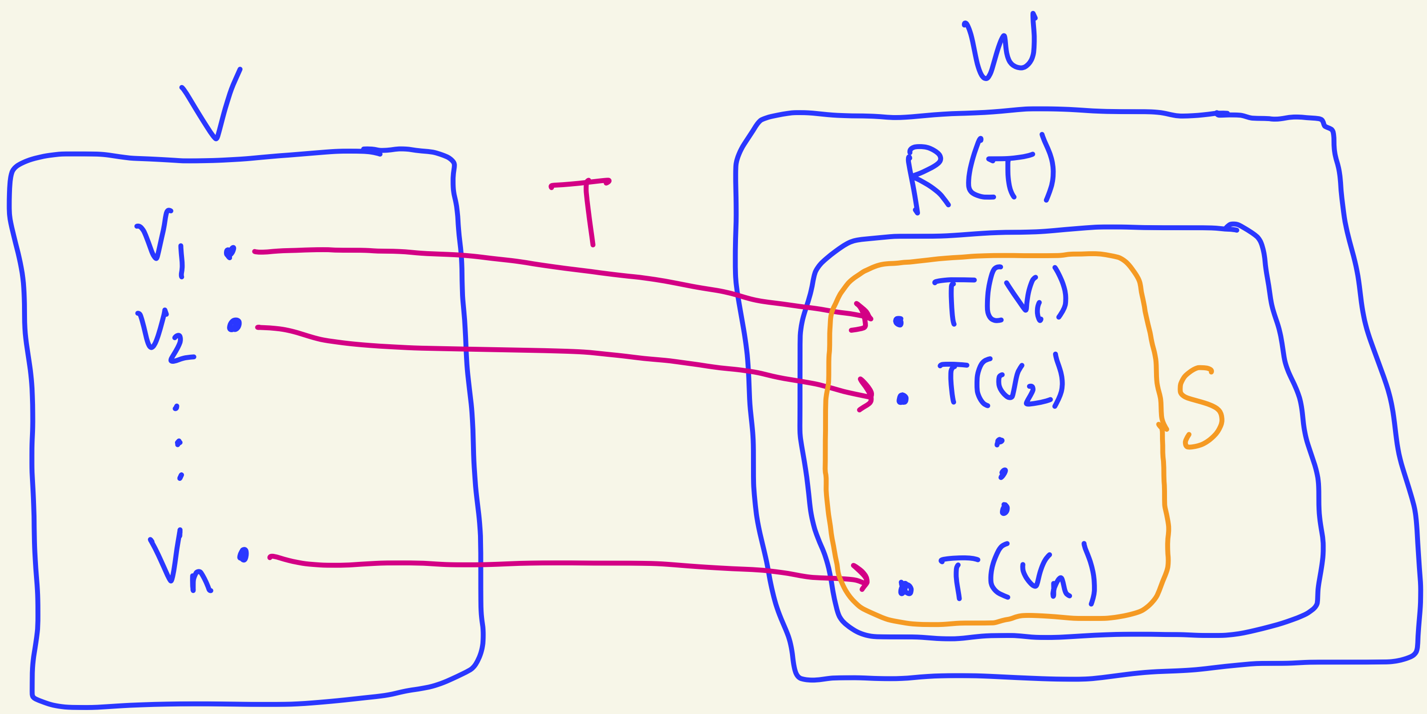
$$= \begin{pmatrix} 2 \\ 0.43-2i \end{pmatrix}$$

⑤ Let $T: V \to W$ be a linear transformation

Let $v_1, v_2, \dots, v_n \in V$ and $V = \text{span}(\{v_1, v_2, \dots, v_n\})$

Let $S = \text{span}(\{T(v_1), T(v_2), \dots, T(v_n)\})$

We want to show that $S = R(T)$



We break the proof into two parts.

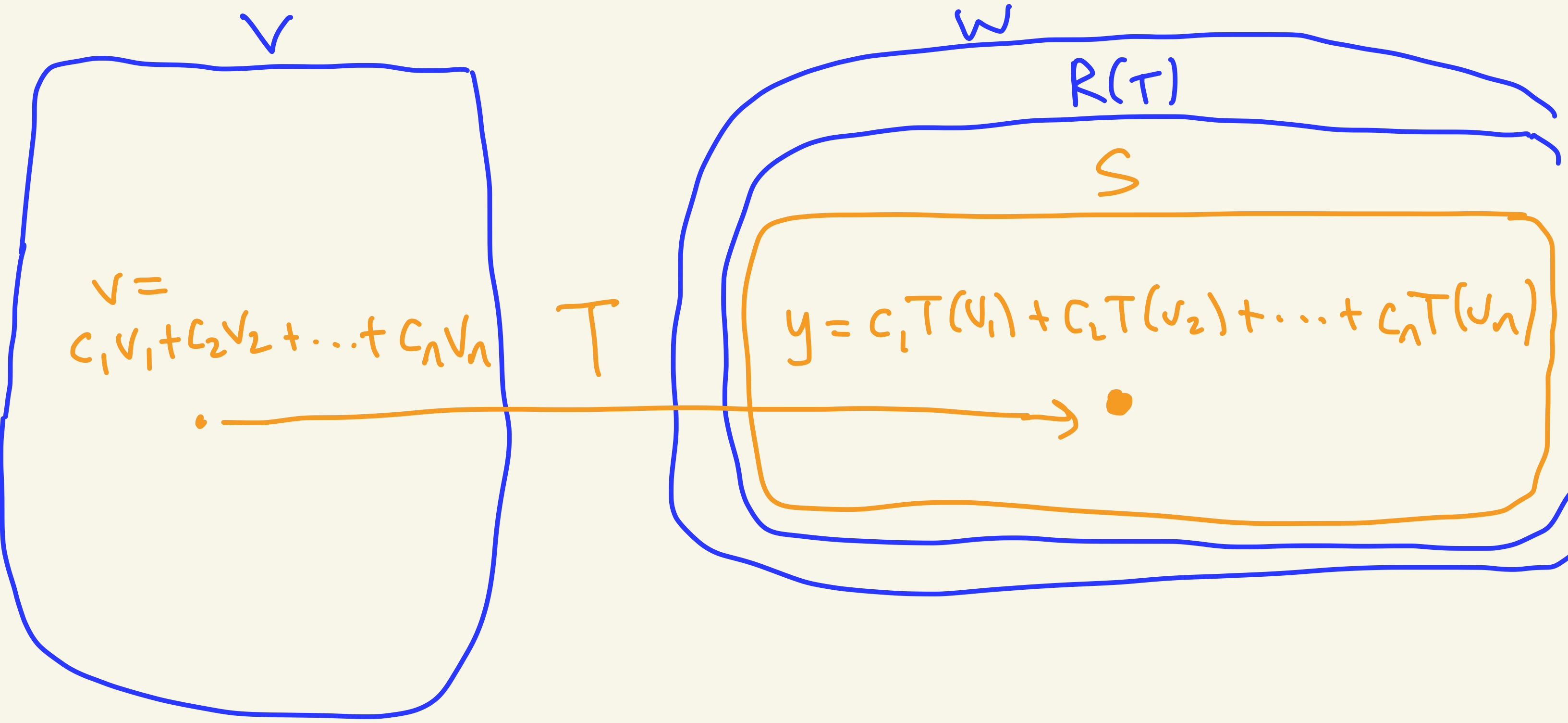
Claim 1: First we show that $S \subseteq R(T)$.

Let $y \in S$. ← we will show this implies that $y \in R(T)$. Thats the goal

Then, $y \in \text{span}(\{T(v_1), T(v_2), \dots, T(v_n)\})$

So there exist scalars $c_1, c_2, \dots, c_n \in F$

where $y = c_1 T(v_1) + c_2 T(v_2) + \dots + c_n T(v_n)$



Then,

$$y = c_1T(v_1)+c_2T(v_2)+\dots+c_nT(v_n)$$
$$= T(c_1v_1+c_2v_2+\dots+c_nv_n) = T(v)$$

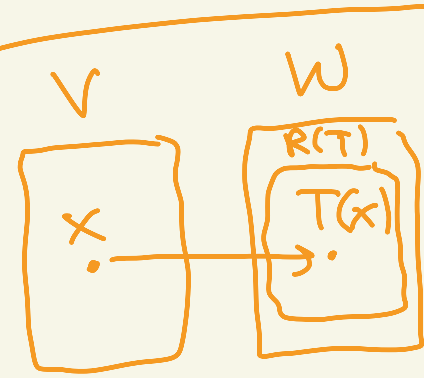
where $v = c_1v_1+c_2v_2+\dots+c_nv_n$

because T is a linear transformation

Because $v_1, v_2, \dots, v_n$ are in V we know that $v = c_1v_1+c_2v_2+\dots+c_nv_n$ is in V.

Thus, $y = T(v) \in R(T)$.

Recall $R(T) = \{T(x) \mid x \in V\}$



So, we have shown claim I, $S \subseteq R(T)$

Claim 2: $R(T) \subseteq S$

Let $y \in R(T)$.

we need to show that this y is also in S. Thats the goal

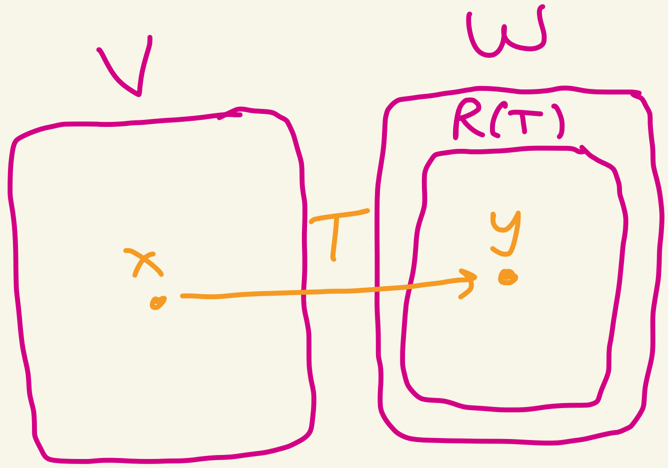
Then by the definition of $R(T)$ we know that $y = T(x)$ where $x \in V$.

Since $x \in V$ and $V = \text{span}(\{v_1, v_2, \ldots, v_n\})$ we know that there exist $c_1, c_2, \ldots, c_n \in F$ where $x = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$.

Then,

$$y = T(x) = T(c_1 v_1 + \ldots + c_n v_n)$$
$$= c_1 T(v_1) + c_2 T(v_2) + \ldots + c_n T(v_n)$$

So, $y \in \underbrace{\text{span}(\{T(v_1), T(v_2), \ldots, T(v_n)\})}_{\text{this is } S}$



So, $y \in S$.

Thus, $R(T) \subseteq S$.

So we have shown claim 2 is true.

By claim 1 and claim 2 we have that $R(T) = S$. $\blacksquare$

⑥ (a) We want to prove the following:

$$\boxed{T \text{ is one-to-one iff } N(T) = \{0_v\}}$$

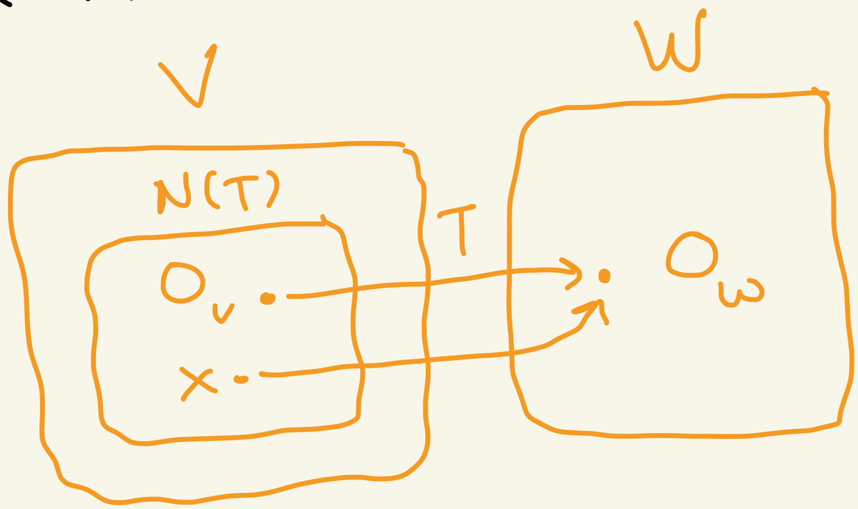
($\Rightarrow$) Suppose that T is one-to-one.
We will show that this implies that $N(T) = \{0_v\}$

Because T is a linear transformation by a theorem in class (and in this HW) we know that $T(0_v) = 0_w$.

Thus, $0_v \in N(T)$

So, $\{0_v\} \subseteq N(T)$.

Why are these two sets equal?



Suppose $x \in N(T)$

Then $T(x) = 0_w$.

But $0_w = T(0_v)$.

Thus, $T(x) = 0_w = T(0_v)$.

By T is one-to-one and so since $T(x) = T(0_v)$ we know that $x = 0_v$.

We have shown that if $x \in N(T)$, then $x = 0_V$.

Thus, $N(T) \subseteq \{0_V\}$.

Since $\{0_V\} \subseteq N(T)$ and $N(T) \subseteq \{0_V\}$ we know that $N(T) = \{0_V\}$.

($\Leftarrow$) Suppose that $N(T) = \{0_V\}$.

Let's show that this implies that T is one-to-one.

Suppose that $T(x) = T(y)$ where $x, y \in V$.

We must show that this implies that $x = y$ to show that T is one-to-one.

Since $T(x) = T(y)$ we have that $T(x) - T(y) = 0$

Thus, $T(x-y) = 0_W$ since T is linear

So, $x - y \in N(T)$.

Thus, $x - y = 0_V$. here we used our assumption that $N(T) = \{0_V\}$

So, $x = y$.

The above argument shows that if $N(T) = \{0_V\}$ then this implies that T is one-to-one.

⑥ (b) The assumption for this problem is that V and W are finite dimesional and that $\dim(V) = \dim(W)$.

$T: V \to W$ is a linear transformation.

$\boxed{\text{Prove: } T \text{ is one-to-one iff } T \text{ is onto}}$

($\Rightarrow$) Suppose that T is one-to-one.

To show that this implies that T is onto we must show that $R(T) = W$.

Because T is one-to-one we know from problem 6(a) that $N(T) = \{0_V\}$ and so $\dim(N(T)) = 0$.

Also above we have the assumption for this problem that $\dim(V) = \dim(W)$.

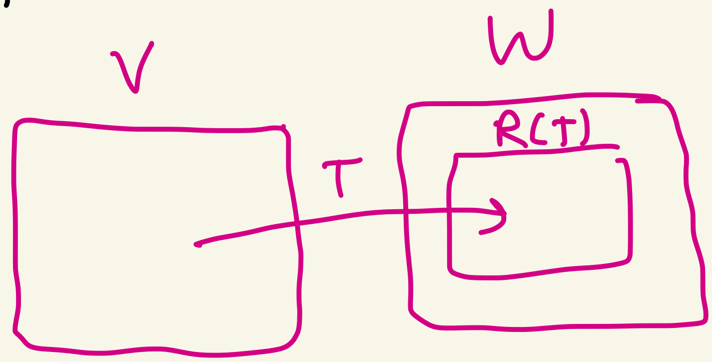
Thus,

rank-nullity theorem

$$\dim(W) = \dim(V) = \dim(N(T)) + \dim(R(T))$$
$$= 0 + \dim(R(T))$$
$$= \dim(R(T)).$$

Thus, $\dim(W) = \dim(R(T))$.

Since $R(T)$ is a subspace of W and $\dim(R(T)) = \dim(W)$ we have from a theorem in class that $W = R(T)$.



Thus, T is onto.

($\Leftarrow$) Now suppose that T is onto.

We will show that T is one-to-one by showing that $N(T) = \{0_V\}$. This will use problem 6(a)

Since T is onto we know $R(T) = W$.

Thus, $\dim(R(T)) = \dim(W)$.

By the assumptions of this problem we know that $\dim(V) = \dim(W)$.

Thus,

rank-nullity thm

$$\dim(W) = \dim(V) = \dim(N(T)) + \dim(R(T))$$

these are equal

So,

$$\dim(W) = \dim(N(T)) + \dim(W)$$

Thus, $\dim(N(T)) = 0$.

So, $N(T) = \{0_V\}$.

Thus, by problem 6(a), T is one-to-one.

$\boxed{/\!/}$

---

⑥ (c) Suppose that T is one-to-one and onto.

Since T is one-to-one we know that $N(T) = \{0_V\}$ and so $\dim(N(T)) = 0$ by problem 6(a).

Since T is onto, we know $R(T) = W$ and so $\dim(R(T)) = \dim(W)$.

Thus, by the rank-nullity theorem we have

$$\begin{aligned} \dim(V) &= \dim(N(T)) + \dim(R(T)) \\ &= 0 + \dim(W) \\ &= \dim(W). \end{aligned}$$

$\boxed{/\!/}$

⑦ (a) [Method 1]

Suppose that $\dim(V) < \dim(W)$.

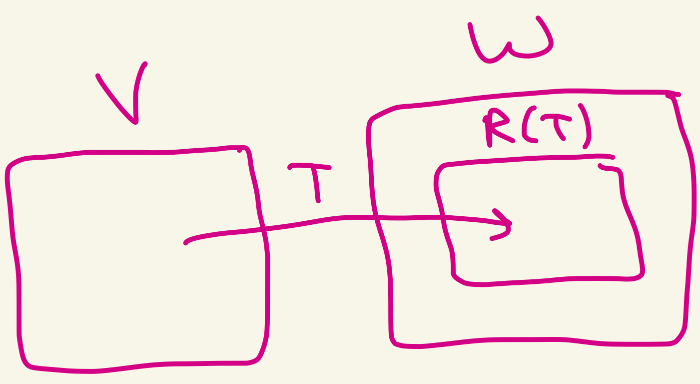
Then,

$$\begin{aligned} \dim(R(T)) &\le \dim(N(T)) + \dim(R(T)) \\ &= \dim(V) \quad \text{rank-nullity thm} \\ &< \dim(W) \end{aligned}$$

Thus,

$$\dim(R(T)) < \dim(W).$$

Since $R(T)$ is a subspace of W and $\dim(R(T)) < \dim(W)$ we must have $R(T) \neq W$.



Thus, T is not onto.

⑦ (a) [Method 2]

Let's prove the contrapositive:

"If T is onto, then $\dim(V) \geq \dim(W)$"

Suppose T is onto.

Then, $R(T) = W$.

Then, $\dim(R(T)) = \dim(W)$.

Thus,

$$\begin{aligned} \dim(V) &\overset{\text{rank-nullity thm}}{=} \dim(N(T)) + \dim(R(T)) \\ &\geq \dim(R(T)) \\ &= \dim(W). \end{aligned}$$

$\dim(N(T)) \geq 0$

Thus, $\dim(V) \geq \dim(W)$. ∎

7 (b)

Let's prove the contrapositive:

"If T is one-to-one, then $\dim(V) \leq \dim(W)$."

Suppose T is one-to-one.

Then by problem 6(a) we know that $N(T) = \{0_V\}$ and so $\dim(N(T)) = 0$.

Because $R(T)$ is a subspace of W we know that $\dim(R(T)) \leq \dim(W)$.

Thus,

rank-nullity theorem

$$
\begin{aligned}
\dim(V) &= \dim(N(T)) + \dim(R(T)) \\
&= 0 + \dim(R(T)) \\
&= \dim(R(T)) \\
&\leq \dim(W).
\end{aligned}
$$

So, $\dim(V) \leq \dim(W)$.