Homework 6 Solutions

① Let $z = a + ib$ and $w = c + id$, where $a, b, c, d \in \mathbb{R}$,

(a) $\overline{\overline{z}} = \overline{\overline{(a+ib)}} = \overline{a - ib} = a + ib = z$

(b) $\overline{z+w} = \overline{(a+c)+i(b+d)} = (a+c) - i(b+d) = (a-ib)+(c-id)$
$= \overline{z} + \overline{w}$

(c) $\overline{zw} = \overline{(a+ib)(c+id)} = \overline{(ac-bd)+i(bc+ad)}$
$= (ac-bd) - i(bc+ad) = ac - bd - ibc - iad$
$= (a-ib)(c-id) = \overline{z}\,\overline{w}$

(d) $\overline{\left(\frac{z}{w}\right)} = \overline{\frac{a+ib}{c+id}} = \overline{\frac{(a+ib)(c-id)}{(c+id)(c-id)}} = \overline{\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}}$
$= \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$
$= \frac{ac+bd}{c^2+d^2} - i\frac{bc-ad}{c^2+d^2}$
$= \frac{ac+bd}{c^2+d^2} + i\frac{ad-bc}{c^2+d^2}$

and $\frac{\overline{z}}{\overline{w}} = \frac{a-ib}{c-id} = \frac{(a-ib)(c+id)}{(c-id)(c+id)}$
$= \frac{(ac+bd)+i(ad-bc)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{ad-bc}{c^2+d^2}$

(e) $|z|^2 = \left(\sqrt{a^2+b^2}\right)^2 = a^2+b^2$
$= (a+ib)(a-ib) = z\overline{z}$

(f) $z\overline{z} = (a+ib)(a-ib) = a^2+b^2 \in \mathbb{R}$
and $z\overline{z} = a^2+b^2 \geq 0$.
And $z\overline{z} = a^2+b^2 = 0$ iff $a^2+b^2=0$ iff
$a=0$ and $b=0$ iff $z = a+ib = 0$.

② (a) By thm in HW, an orthonormal set β linearly independent. Since β has 2 elements and $\dim(\mathbb{R}^2) = 2$ we just have to check that β is an ~~orthogonal~~ orthonormal set and that will imply that β is also a basis.

We verify that

$$\left\langle \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\rangle = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 0,$$

So we have an orthogonal set.

Also

$$\left\| \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \right\| = \sqrt{\left\langle \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \right\rangle} = \sqrt{\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}} \cdot \frac{-1}{\sqrt{2}}}$$
$$= \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

and

$$\left\| \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\| = \sqrt{\left\langle \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\rangle} = \sqrt{\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}}$$
$$= \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

So,

$$W = (3,7) = \frac{\left\langle (3,7), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \right\rangle}{\left\| \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \right\|} \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) + \frac{\left\langle (3,7), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\rangle}{\left\| \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\|} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= \left(\frac{\frac{3}{\sqrt{2}} - \frac{7}{\sqrt{2}}}{1}\right) \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) + \frac{\frac{3}{\sqrt{2}} + \frac{7}{\sqrt{2}}}{1} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= \frac{-4}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) + \frac{10}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

(b) As in part a, since β has 3 elements and $\dim(\mathbb{R}^3) = 3$ we just need to verify that β is an orthonormal set to get a basis.

Let's do this.

$$\left\langle \left(\tfrac{2}{3}, \tfrac{-2}{3}, \tfrac{1}{3}\right), \left(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{-2}{3}\right) \right\rangle = \tfrac{2}{3}\cdot\tfrac{2}{3} - \tfrac{2}{3}\cdot\tfrac{1}{3} + \tfrac{1}{3}\cdot\left(\tfrac{-2}{3}\right) = 0$$

$$\left\langle \left(\tfrac{2}{3}, \tfrac{-2}{3}, \tfrac{1}{3}\right), \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right) \right\rangle = \tfrac{2}{3}\cdot\tfrac{1}{3} + \left(\tfrac{-2}{3}\right)\left(\tfrac{2}{3}\right) + \tfrac{1}{3}\cdot\tfrac{2}{3} = 0$$

$$\left\langle \left(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{-2}{3}\right), \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right) \right\rangle = \tfrac{2}{3}\cdot\tfrac{1}{3} + \tfrac{1}{3}\cdot\tfrac{2}{3} - \tfrac{2}{3}\cdot\tfrac{2}{3} = 0$$

So we have an orthogonal set.

And also,

$$\left\| \left(\tfrac{2}{3}, \tfrac{-2}{3}, \tfrac{1}{3}\right) \right\| = \sqrt{\left(\tfrac{2}{3}\right)\left(\tfrac{2}{3}\right) + \left(\tfrac{-2}{3}\right)\left(\tfrac{-2}{3}\right) + \left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right)} = \sqrt{\tfrac{9}{9}} = 1$$

$$\left\| \left(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{-2}{3}\right) \right\| = \sqrt{\left(\tfrac{2}{3}\right)\left(\tfrac{2}{3}\right) + \left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) + \left(\tfrac{-2}{3}\right)\left(\tfrac{-2}{3}\right)} = \sqrt{\tfrac{9}{9}} = 1$$

$$\left\| \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right) \right\| = \sqrt{\left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) + \left(\tfrac{2}{3}\right)\left(\tfrac{2}{3}\right) + \left(\tfrac{2}{3}\right)\left(\tfrac{2}{3}\right)} = \sqrt{\tfrac{9}{9}} = 1$$

So we have an orthonormal set.

Thus,

$$w = (-1,0,2) = \frac{\left\langle w, \left(\tfrac{2}{3}, \tfrac{-2}{3}, \tfrac{1}{3}\right) \right\rangle}{\left\| \left(\tfrac{2}{3}, \tfrac{-2}{3}, \tfrac{1}{3}\right) \right\|} \left(\tfrac{2}{3}, \tfrac{-2}{3}, \tfrac{1}{3}\right) + \frac{\left\langle w, \left(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{-2}{3}\right) \right\rangle}{\left\| \left(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{-2}{3}\right) \right\|} \left(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{-2}{3}\right)$$

$$+ \frac{\left\langle w, \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right) \right\rangle}{\left\| \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right) \right\|} \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right) =$$

$$= \frac{-\frac{2}{3}+0+\frac{2}{3}}{1}\left(\frac{2}{3},\frac{-2}{3},\frac{1}{3}\right)+\frac{\frac{-2}{3}+0-\frac{4}{3}}{1}\left(\frac{2}{3},\frac{1}{3},\frac{-2}{3}\right)$$

$$+\frac{-1/3+0+4/3}{1}\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$$

$$= 0\cdot\left(\frac{2}{3},\frac{-2}{3},\frac{1}{3}\right) - 2\cdot\left(\frac{2}{3},\frac{1}{3},\frac{-2}{3}\right) + 1\cdot\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$$

---

3 (a) Let $w_1=(1,1,1), w_2=(-1,1,0), w_3=(1,2,1)$.

We now do Gram-Schmidt.

$$v_1 = w_1 = \boxed{(1,1,1)}$$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2}v_1 = (-1,1,0) - \frac{(-1+1)}{(\sqrt{1^2+1^2+1^2})^2}(1,1,1)$$

$$= (-1,1,0) - 0 = \boxed{(-1,1,0)}$$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2}v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2}v_2$$

$$= (1,2,1) - \frac{1+2+1}{(\sqrt{1^2+1^2+1^2})^2}(1,1,1) - \frac{-1+2+0}{(\sqrt{(-1)^2+1^2+0^2})^2}(-1,1,0)$$

$$= (1,2,1) - \frac{4}{3}(1,1,1) - \frac{1}{2}(-1,1,0)$$

$$= \boxed{\left(\frac{1}{6},\frac{1}{6},-\frac{1}{3}\right)}$$

$\{v_1, v_2, v_3\}$ is an orthogonal basis.

Now we normalize.

Let $v_1' = \frac{v_1}{\|v_1\|} = \frac{(1,1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$,

$$V_2' = \frac{V_2}{\|V_2\|} = \frac{(-1,1,0)}{\sqrt{2}} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$V_3' = \frac{V_3}{\|V_3\|} = \frac{\left(\frac{1}{6},\frac{1}{6},-\frac{1}{3}\right)}{\sqrt{\left(\frac{1}{6}\right)^2+\left(\frac{1}{6}\right)^2+\left(-\frac{1}{3}\right)^2}} = \frac{\left(\frac{1}{6},\frac{1}{6},-\frac{1}{3}\right)}{\sqrt{\frac{1}{6}}} = \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}\right).$$

Then $\{v_1', v_2', v_3'\}$ is an orthonormal basis for $\mathbb{R}^3$.

(b) Let $w_1 = (1,0,0)$, $w_2 = (3,7,-2)$, $w_3 = (0,4,1)$.

We now do Gram-Schmidt.

$$V_1 = W_1 = \boxed{(1,0,0)}$$

$$V_2 = W_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = (3,7,2) - \frac{3}{1}(1,0,0) = \boxed{(0,7,2)}$$

$$V_3 = W_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2$$

$$= (0,4,1) - \frac{0}{1}(1,0,0) - \frac{0+28+2}{\left(\sqrt{0^2+7^2+2^2}\right)^2}(0,7,2)$$

$$= (0,4,1) - \frac{30}{53}(0,7,2)$$

$$= \boxed{\left(0, \frac{2}{53}, \frac{-7}{53}\right)}$$

Then $\{v_1, v_2, v_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

To make it orthonormal we have to normalize the vectors.

Let $$v_1' = \frac{v_1}{\|v_1\|} = \frac{(1,0,0)}{1} = (1,0,0)$$

$$v_2' = \frac{v_2}{\|v_2\|} = \frac{(0,7,2)}{\sqrt{0^2+7^2+2^2}} = \left(0, \frac{7}{\sqrt{53}}, \frac{2}{\sqrt{53}}\right)$$

$$v_3' = \frac{v_3}{\|v_3\|} = \frac{\left(0,\frac{2}{53},-\frac{7}{53}\right)}{\sqrt{0^2+\left(\frac{2}{53}\right)^2+\left(\frac{-7}{53}\right)^2}}$$

$$= \sqrt{53}\left(0, \frac{2}{53}, -\frac{7}{53}\right)$$

$$= \left(0, \frac{2\sqrt{53}}{53}, -\frac{7\sqrt{53}}{53}\right).$$

Then, $\{v_1', v_2', v_3'\}$ is an orthonormal basis for $\mathbb{R}^3$.

(c) Let $w_1 = (0,2,1,0)$, $w_2 = (1,-1,0,0)$,
$w_3 = (1,2,0,-1)$, $w_4 = (1,0,0,1)$

We now apply Gram Schmidt.

$$v_1 = w_1 = \boxed{(0,2,1,0)}$$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = (1,-1,0,0) - \frac{0-2+0+0}{\left(\sqrt{0^2+2^2+1^2+0^2}\right)^2}(0,2,1,0)$$

$$= (1,-1,0,0) - \frac{-2}{5}(0,2,1,0) = \boxed{\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)}$$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2 = (1,2,0,-1) - \frac{0+4+0+0}{\sqrt{0^2+2^2+1^2+0^2}}(0,2,1,0)$$

$$- \frac{1-\frac{2}{5}+0+0}{\left(\sqrt{1^2+(-\frac{1}{5})^2+(\frac{2}{5})^2+0^2}\right)^2}\left(1,-\frac{1}{5},\frac{2}{5},0\right)$$

$\frac{1}{2}$

$$= (1,2,0,-1) - \frac{4}{5}(0,2,1,0) - \frac{3/5}{(6/5)}\left(1,-\frac{1}{5},\frac{2}{5},0\right)$$

$$= \left(1-0-\frac{1}{2},\ 2-\frac{8}{5}+\frac{1}{10},\ 0-\frac{4}{5}-\frac{1}{5},\ -1-0-0\right)$$

$$= \boxed{\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)}$$

$$v_4 = w_4 - \frac{\langle w_4, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_4, v_2\rangle}{\|v_2\|^2} v_2 - \frac{\langle w_4, v_3\rangle}{\|v_3\|^2} v_3$$

$$= (1,0,0,1) - \frac{0}{\|v_1\|^2} v_1 - \frac{1+0+0+0}{\left(\sqrt{1^2+(-\frac{1}{5})^2+(\frac{2}{5})^2+0^2}\right)^2}\left(1,-\frac{1}{5},\frac{2}{5},0\right)$$

$$- \frac{1/2+0+0-1}{\left(\sqrt{(\frac{1}{2})^2+(\frac{1}{2})^2+(-1)^2+(-1)^2}\right)^2}\left(\frac{1}{2},\frac{1}{2},-1,-1\right)$$

$5/6$ $\quad$ $-1/5$

$$= (1,0,0,1) - \frac{1}{(6/5)}\left(1,-\frac{1}{5},\frac{2}{5},0\right) - \frac{-1/2}{5/2}\left(\frac{1}{2},\frac{1}{2},-1,-1\right)$$

$$= \left(1-\frac{5}{6}+\frac{1}{10},\ 0+\frac{1}{6}+\frac{1}{10},\ 0-\frac{1}{3}-\frac{1}{5},\ 1-0-1/5\right) = \boxed{\left(\frac{4}{15}, \frac{4}{15}, \frac{-8}{15}, \frac{4}{5}\right)}$$

Then $\{v_1, v_2, v_3, v_4\}$ is an orthogonal basis for $\mathbb{R}^4$. We now normalize.

$$v_1' = \frac{v_1}{\|v_1\|} = \frac{(0,2,1,0)}{\sqrt{0^2+2^2+1^2+0^2}} = \frac{(0,2,1,0)}{\sqrt{5}} = \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$$

$$v_2' = \frac{v_2}{\|v_2\|} = \frac{(1, -1/5, 2/5, 0)}{\sqrt{1^2+(-\frac{1}{5})^2+(\frac{2}{5})^2+0^2}} = \frac{\sqrt{5}}{\sqrt{6}}\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

$$= \left(\frac{\sqrt{5}}{\sqrt{6}}, \frac{-\sqrt{5}}{5\sqrt{6}}, \frac{2\sqrt{5}}{5\sqrt{6}}, 0\right)$$

$$v_3' = \frac{v_3}{\|v_3\|} = \frac{(1/2, 1/2, -1, -1)}{\sqrt{(\frac{1}{2})^2+(\frac{1}{2})^2+(-1)^2+(-1)^2}} = \frac{\sqrt{2}}{\sqrt{5}}\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)$$

$$= \left(\frac{\sqrt{2}}{2\sqrt{5}}, \frac{\sqrt{2}}{2\sqrt{5}}, \frac{-\sqrt{2}}{\sqrt{5}}, \frac{-\sqrt{2}}{\sqrt{5}}\right)$$

$$v_4' = \frac{v_4}{\|v_4\|} = \frac{\left(\frac{4}{15}, \frac{4}{15}, \frac{-8}{15}, \frac{4}{5}\right)}{\sqrt{(\frac{4}{15})^2+(\frac{4}{15})^2+(\frac{-8}{15})^2+(\frac{4}{15})^2}} = \frac{\sqrt{15}}{4}\left(\frac{4}{15}, \frac{4}{15}, \frac{-8}{15}, \frac{4}{15}\right)$$

$$= \left(\frac{\sqrt{15}}{15}, \frac{\sqrt{15}}{15}, -\frac{2\sqrt{15}}{15}, \frac{\sqrt{15}}{15}\right).$$

Then $\{v_1', v_2', v_3', v_4'\}$ is an orthonormal basis for $\mathbb{R}^4$.

(4) Let $x, y, z \in V$ and $c \in F$.

(a) We have that $\langle x, y+z \rangle = \overline{\langle y+z, x \rangle} =$
$= \overline{\langle y, x \rangle + \langle z, x \rangle} = \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle}$
$= \langle x, y \rangle + \langle x, z \rangle$.

(b) We have that $\langle x, cy \rangle = \overline{\langle cy, x \rangle} =$
$= \overline{c \langle y, x \rangle} = \bar{c}\, \overline{\langle y, x \rangle} = \bar{c} \langle x, y \rangle$.

(c) Note that
$\langle \vec{0}, \vec{0} \rangle = \langle \vec{0} + \vec{0}, \vec{0} \rangle = \langle \vec{0}, \vec{0} \rangle + \langle \vec{0}, \vec{0} \rangle$
by the def of inner product. So, by adding
$-\langle \vec{0}, \vec{0} \rangle$ + both sides we get. That
$0 = \langle \vec{0}, \vec{0} \rangle$.
Since $\langle x, x \rangle > 0$ if $x \neq \vec{0}$ (by def of inner product), This proves the result.

(5) Let $x, y \in V$ and $c \in F$.

(a) We have that $\|x\|^2 = \left(\sqrt{\langle x,x\rangle}\right)^2 = \langle x,x\rangle$.

(b) We have that

$$\|cx\| = \sqrt{\langle cx, cx\rangle} = \sqrt{c\langle x, cx\rangle} = \sqrt{c\bar{c}\langle x,x\rangle}$$

$$= \sqrt{c\bar{c}}\sqrt{\langle x,x\rangle} = \sqrt{|c|^2}\,\|x\| = |c|\cdot\|x\|,$$

(c) We have that $\|x\| = \sqrt{\langle x,x\rangle} \geq 0$ by the def of inner product and problem 4(c). By 4(c), $\|x\| = \sqrt{\langle x,x\rangle} = 0$ iff $\langle x,x\rangle = 0$ iff $x = \vec{0}$.

(d) Let $\{v_1, v_2, \ldots, v_n\}$ be an orthogonal set in V and $a_1, a_2, \ldots, a_n \in F$. Then

$$\left\|\sum_{i=1}^n a_i v_i\right\|^2 = \left\langle \sum_{i=1}^n a_i v_i, \sum_{j=1}^n a_j v_j\right\rangle = \sum_{i=1}^n \left\langle a_i v_i, \sum_{j=1}^n a_j v_j\right\rangle$$

$$= \sum_{i=1}^n a_i \left\langle v_i, \sum_{j=1}^n a_j v_j\right\rangle = \sum_{i=1}^n a_i \sum_{j=1}^n \langle v_i, a_j v_j\rangle$$

$$= \sum_{i=1}^n \sum_{j=1}^n a_i \overline{a_j}\langle v_i, v_j\rangle = a_1\overline{a_1}\langle v_1, v_1\rangle + a_2\overline{a_2}\langle v_2, v_2\rangle + \cdots + a_n\overline{a_n}\langle v_n, v_n\rangle$$

since we have orthogonality

$$\boxed{\langle v_i, v_j\rangle = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}}$$

$$= \sum_{i=1}^n |a_i|^2 \|v_i\|^2$$

⑥ (a) Let $v \in V$. Then

$$\langle \vec{0}, v \rangle = \langle \vec{0} + \vec{0}, v \rangle = \langle \vec{0}, v \rangle + \langle \vec{0}, v \rangle.$$

Add $-\langle \vec{0}, v \rangle$ to both sides to get $0 = \langle \vec{0}, v \rangle$. Thus, we also have, $\langle v, \vec{0} \rangle = \overline{\langle \vec{0}, v \rangle} = \overline{0} = 0$.

(b) Let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthogonal set of vectors. Then $v_j \neq \vec{0}$ for all j.

Suppose that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \vec{0}$$

for some $c_1, c_2, \ldots, c_n$. We now show that $c_i = 0$. Let $1 \leq i \leq n$.

This will show linear independence.

We have that

$$\langle c_1 v_1 + c_2 v_2 + \cdots + c_n v_n, v_i \rangle = \underbrace{\langle \vec{0}, v_i \rangle}_{0 \text{ by part a}}.$$

So,

$$\langle c_1 v_1, v_i \rangle + \langle c_2 v_2, v_i \rangle + \cdots + \langle c_i v_i, v_i \rangle + \cdots + \langle c_n v_n, v_i \rangle = 0.$$

Thus,

$$c_1 \langle v_1, v_i \rangle + c_2 \langle v_2, v_i \rangle + \cdots + c_i \langle v_i, v_i \rangle + \cdots + c_n \langle v_n, v_i \rangle = 0.$$

Since S is an orthogonal set $\langle v_j, v_i \rangle = 0$ if $j \neq i$. Hence we get $c_i \langle v_i, v_i \rangle = 0$. Since $v_i \neq \vec{0}$, we have $\langle v_i, v_i \rangle \neq \vec{0}$. Thus, $c_i = \frac{0}{\langle v_i, v_i \rangle} = 0$. ∎

(7) $W = \operatorname{span}\left(\left\{\begin{pmatrix}\text{[redacted]}\\0\\0\end{pmatrix}\right\}\right) = \left\{\begin{pmatrix}x\\0\\0\end{pmatrix} \,\middle|\, x \in \mathbb{R}\right\}$

$$W^\perp = \left\{\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3 \,\middle|\, \left\langle \begin{pmatrix}a\\b\\c\end{pmatrix}, \begin{pmatrix}x\\0\\0\end{pmatrix}\right\rangle = 0 \text{ for all } x \in \mathbb{R}\right\}$$

$$= \left\{\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3 \,\middle|\, ax + b0 + c0 = 0 \text{ for all } x \in \mathbb{R}\right\}$$

$$= \left\{\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3 \,\middle|\, ax = 0 \text{ for all } x \in \mathbb{R}\right\}$$

$$= \left\{\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3 \,\middle|\, a = 0,\ b, c \in \mathbb{R}\right\}$$

$$= \left\{\begin{pmatrix}0\\b\\c\end{pmatrix} \,\middle|\, b, c \in \mathbb{R}\right\}$$

$$= \left\{ b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix} \,\middle|\, b, c \in \mathbb{R}\right\}$$

$$= \operatorname{span}\left(\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\right),$$

Note that $\dim(W^\perp) = 2$.

---

(8) (a) Recall that $W^\perp = \{x \in V \mid \langle x, w\rangle = 0 \text{ for all } w \in W\}$

Note that $\langle \vec{0}, w\rangle = 0$ for all $w \in W$ by problem 6(a).

Hence $\vec{0} \in W^\perp$,

Let $x, y \in W^\perp$ and $\alpha \in F$.

Then $\langle x, w\rangle = 0$ and $\langle y, w\rangle = 0$ for all $w \in W$.

Thus, $\langle x+y, w\rangle = \langle x, w\rangle + \langle y, w\rangle = 0 + 0 = 0$ for all $w \in W$. And $\langle \alpha x, w\rangle = \alpha\langle x, w\rangle = \alpha 0 = 0$ for all $w \in W$. So, $x+y \in W^\perp$ and $\alpha x \in W^\perp$.

So, $W^\perp$ is a subspace of V.

(b) $\{\vec{0}\}^\perp = \{x \in V \mid \langle x, \vec{0} \rangle = 0\} = V$.
by problem 6(a).

(c) $V^\perp = \{x \in V \mid \langle x, y \rangle = 0 \text{ for all } y \in V\}$
Note that $\langle x, x \rangle = 0$ iff $x = \vec{0}$ and
$\langle \vec{0}, y \rangle = 0$ for all $y \in V$. Hence
$V^\perp = \{\vec{0}\}$.

(d) Suppose that $W_1 \subseteq W_2$. Let $x \in W_2^\perp$.
Then $\langle x, w \rangle = 0$ for all $w \in W_2$.
Since $W_1 \subseteq W_2$ this implies that
$\langle x, w \rangle = 0$ for all $w \in W_2$
Thus, $x \in W_1^\perp$. So, $W_2^\perp \subseteq W_1^\perp$.

9 ($\Rightarrow$) Suppose that $V = W_1 \oplus W_2$.
By def, $V = W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1 \text{ and } w_2 \in W_2\}$
and $W_1 \cap W_2 = \{\vec{0}\}$.
Let $x \in V$. Then $x = w_1 + w_2$ where $w_1 \in W_1$ and
$w_2 \in W_2$. Suppose $x = w_1' + w_2'$ for some other
$w_1' \in W_1$ and $w_2' \in W_2$. Then $w_1 + w_2 = w_1' + w_2'$.

So, $w_1 - w_1' = w_2 - w_2'$. Hence, $w_1 - w_1' \in W_1 \cap W_2 = \{\vec{0}\}$ and $w_2 - w_2' \in W_1 \cap W_2 = \{\vec{0}\}$.

So, $w_1 - w_1' = \vec{0}$ and $w_2 - w_2' = \vec{0}$.

So, $w_1 = w_1'$ and $w_2 = w_2'$. So the expression $x = w_1 + w_2$ is unique.

($\Leftarrow$) Suppose every vector $x \in V$ can be expressed uniquely in the form $x = w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$.

So, we get that $V = W_1 + W_2$.

Let's show that $W_1 \cap W_2 = \{\vec{0}\}$.

Let $y \in W_1 \cap W_2$. Then,

$$\underset{W_1}{\underset{\uparrow}{y}} = y + \vec{0} = \vec{0} + y$$

$y = \underset{\substack{\uparrow \\ W_1}}{y} + \underset{\substack{\uparrow \\ W_2}}{\vec{0}} = \underset{\substack{\uparrow \\ W_1}}{\vec{0}} + \underset{\substack{\uparrow \\ W_2}}{y}$. By the uniqueness assumption, $y = \vec{0}$.

So, $W_1 \cap W_2 \subseteq \{\vec{0}\}$.

Since W_1 and W_2 are subspaces $\vec{0} \in W_1$ and $\vec{0} \in W_2$. So, $\vec{0} \in W_1 \cap W_2$

$\{\vec{0}\} \subseteq W_1 \cap W_2$.

Hence, $W_1 \cap W_2 = \{\vec{0}\}$.

So,

(10) We show that $V = W \oplus W^\perp$.

Since V is finite-dimensional, so is W. Let $\beta = [w_1, w_2, \dots, w_n]$ be an orthonormal basis for W.

We will show that $V = W_1 + W_2$ and $W_1 \cap W_2 = \{\vec{0}\}$. This shows that $V = W_1 \oplus W_2$ by definition.

Let $x \in V$.

Construct the vector

$$w = \langle x, w_1\rangle w_1 + \langle x, w_2\rangle w_2 + \dots + \langle x, w_n\rangle w_n.$$

Set $w' = x - w$.

Then $x = w + w'$.

By construction, $w \in W$.

Lets show that $w' \in W^\perp$. We have that

$$\langle w, w'\rangle = \langle w, x-w\rangle = \overline{\langle x-w, w\rangle} = \overline{\langle x,w\rangle - \langle w,w\rangle} = \langle w,x\rangle - \langle w,w\rangle$$

$$= \overline{\langle x,w\rangle} - \overline{\langle w,w\rangle}$$

$$= \left\langle \sum_{i=1}^n \langle x,w_i\rangle w_i, x\right\rangle - \left\langle \sum_{i=1}^n \langle x,w_i\rangle w_i, \sum_{j=1}^n \langle x,w_j\rangle w_j\right\rangle$$

$$= \sum_{i=1}^n \langle \langle x,w_i\rangle w_i, x\rangle - \sum_{i=1}^n \left\langle \langle x,w_i\rangle w_i, \sum_{j=1}^n \langle x,w_j\rangle w_j\right\rangle$$

$$= \sum_{i=1}^n \langle x,w_i\rangle \langle w_i, x\rangle - \sum_{i=1}^n \langle x,w_i\rangle \left\langle w_i, \sum_{j=1}^n \langle x,w_j\rangle w_j\right\rangle$$

$$= \sum_{i=1}^n \langle x,w_i\rangle \overline{\langle x, w_i\rangle} - \sum_{i=1}^n \sum_{j=1}^n \langle x,w_i\rangle \langle w_i, \langle x,w_j\rangle w_j\rangle$$

$$= \sum_{i=1}^n \langle x,w_i\rangle \overline{\langle x, w_i\rangle} - \sum_{i=1}^n \sum_{j=1}^n \langle x,w_i\rangle \overline{\langle x,w_j\rangle} \langle w_i, w_j\rangle$$

$$= \sum_{i=1}^n \langle x,w_i\rangle \overline{\langle x, w_i\rangle} - \sum_{i=1}^n \langle x,w_i\rangle \langle x,w_i\rangle = 0.$$

$\boxed{\langle w_i, w_j\rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}}$

So, $w' \in W^\perp$.

Thus, any $x \in V$ can be written in the form $x = w + w'$ where $w \in W$ and $w' \in W^{\perp}$. So, $V = W + W^{\perp}$.

Now we show that $W \cap W^{\perp} = \{\vec{0}\}$ and we are done.

Since W and $W^{\perp}$ are subspaces of V we have that $\vec{0} \in W$ and $\vec{0} \in W^{\perp}$.

So, $\{\vec{0}\} \subseteq W \cap W^{\perp}$.

Now let $y \in W \cap W^{\perp}$. So $y \in W$ and $y \in W^{\perp}$.

Since $y \in W$, we have that $\langle y, z \rangle = 0$ for all $z \in W^{\perp}$.

Since $y \in W^{\perp}$ this implies that $\langle y, y \rangle = 0$.

So, $y = \vec{0}$.

Hence, $W \cap W^{\perp} \subseteq \{\vec{0}\}$.

So, $W \cap W^{\perp} = \{\vec{0}\}$.

Since $V = W + W^{\perp}$ and $W \cap W^{\perp} = \{\vec{0}\}$ we have that $V = W \oplus W^{\perp}$.