HANDOUT

Assumptions about $\mathbb{R}$

We assume these sets exist with their usual properties:

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$ ← set of natural numbers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$ ← set of integers

$$\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z},\ b \neq 0 \right\}$$ ← set of rational numbers

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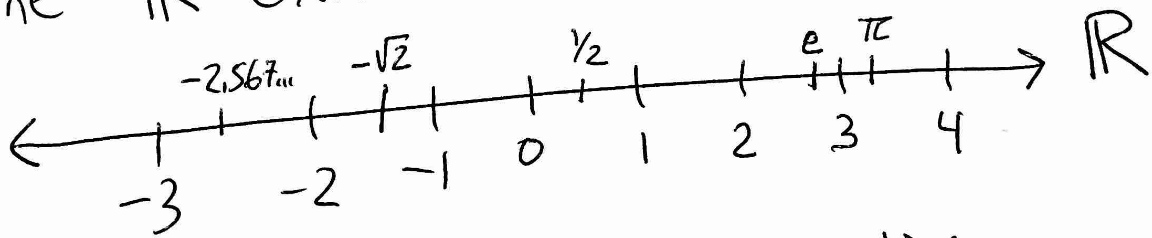
We will assume that the real line $\mathbb{R}$ exists



with all the usual algebraic properties such as

- For every $a, b, c \in \mathbb{R}$

  $a + b = b + a$ $\qquad a + (b + c) = (a + b) + c$

  $ab = ba$ $\qquad a(bc) = (ab)c$

  $a(b + c) = ab + ac$

  $(b + c)a = ba + ca$

- There exists $0 \in \mathbb{R}$ with $0 + a = a + 0 = a$ for all $a \in \mathbb{R}$
- There exists $1 \in \mathbb{R}$ with $1a = a1 = a$ for all $a \in \mathbb{R}$

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- For every $a \in \mathbb{R}$, there exists $-a \in \mathbb{R}$ with $a + (-a) = (-a) + a = 0$
- For every $a \in \mathbb{R}$ with $a \neq 0$, there exists $a^{-1} \in \mathbb{R}$ with $a a^{-1} = a^{-1} a = 1$.

There exist operators $<, >, \leq, \geq$ defined in the usual way. We assume the usual order properties of $\mathbb{R}$, such as:

- Let $a, b, c \in \mathbb{R}$.
  - Either $a = b$, $a < b$, or $b < a$.
  - If $a < b$ and $b < c$, then $a < c$
  - If $a < b$, then $a + c < b + c$.
  - If $c > 0$ and $x < y$, then $cx < cy$.

We assume all the other usual order/ algebraic properties of $\mathbb{R}$, of which the above are only some.

The Completeness Axiom

<u>Def:</u> Let $S \subseteq \mathbb{R}$, S is non-empty.

- We say that S is <u>bounded from above</u> if there exists $b \in \mathbb{R}$ where $s \leq b$ for all $s \in S$. If this is the case, then we call b an <u>upper bound</u> for S.

  Furthermore, if b is an upper bound for S and $b \leq c$ for all other upper bounds c of S, then b is called the <u>least upper bound</u> or <u>supremum</u> of S, and we write

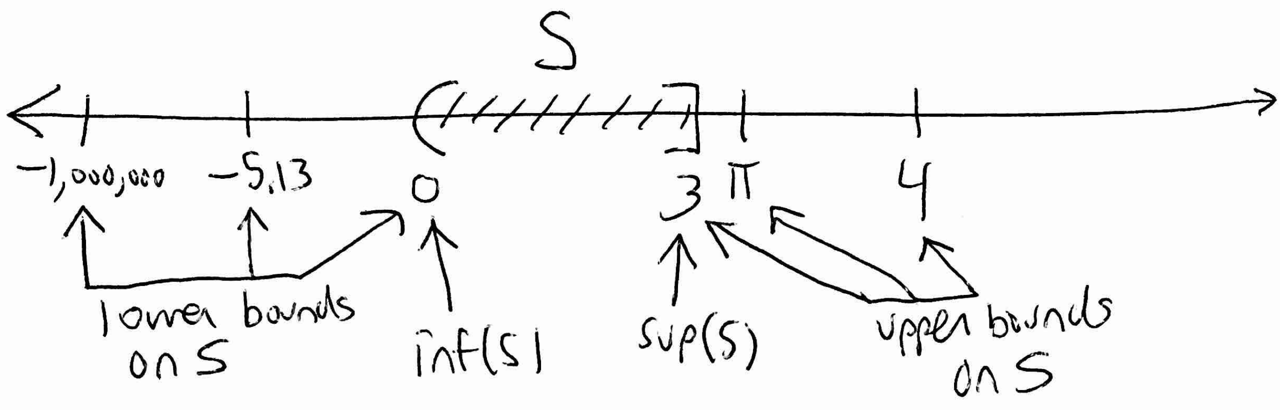
  $\sup(S) = b$.

- We say that S is <u>bounded from below</u> if there exists $b \in \mathbb{R}$ with $b \leq s$ for all $s \in S$. If this is the case, then we call b a <u>lower bound</u> for S.

  Furthermore, if b is a lower bound for S and $c \leq b$ for all other lower bounds of S, then b is called the <u>greatest lower bound</u> or <u>infimum</u> of S and we write

  $\inf(S) = b$.

Ex: $S = (0,3]$



$\inf(S) = 0$
$\sup(S) = 3$

Ex: $S = [1,2] \cup [4,\infty)$

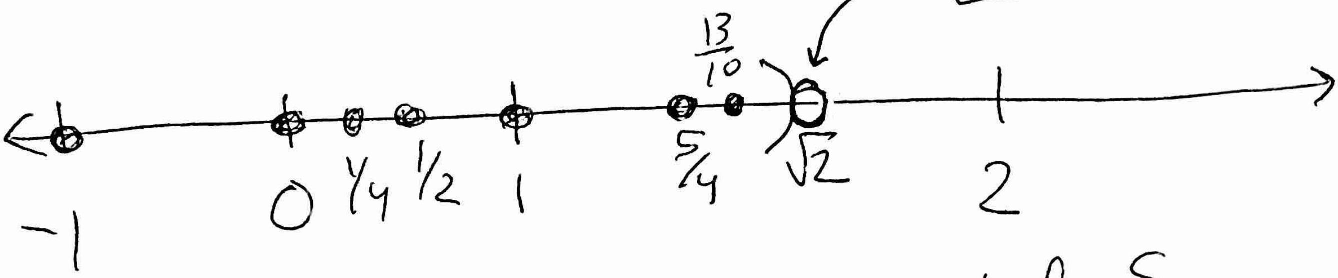
$\inf(S) = 1$
$\sup(S)$ does not exist

HW: If $\inf(S)$ and $\sup(S)$ exist, then they are unique.

Consider the set

$$S = \{x \in \mathbb{Q} \mid x < \sqrt{2}\}$$

There is no rational number here



- $\sqrt{2}$ is the least upper-bound for S.
- However $\sqrt{2} \notin \mathbb{Q}$.
- $\mathbb{Q}$ has "holes" in it.

$\mathbb{R}$ has no "holes." This can be summarized in the following assumption about $\mathbb{R}$.

The completeness Axiom

Let S be a non-empty subset of $\mathbb{R}$. If S is bounded from above, then there exists $b \in \mathbb{R}$ that is the supremum of $\mathbb{R}$.

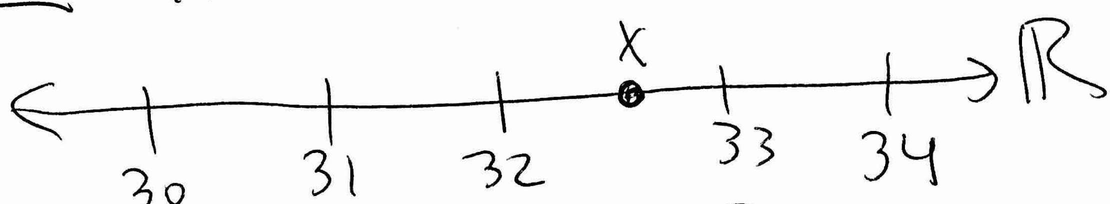
~~Similarly, if $S \subseteq \mathbb{R}$ and $S \neq \emptyset$ and S is bounded from below, then $\exists\, a \in \mathbb{R}$ with $a = \inf(S)$~~

We will show later that this implies that if $S \subseteq \mathbb{R}$, $S \neq \emptyset$, and S is bounded from below, then $\inf(S)$ exists.

Thm: (Archimedean Property)
If $x \in \mathbb{R}$, then there exists $n \in \mathbb{N}$ with $x < n$.

Pf: Suppose this is not true. That is, there exists $x \in \mathbb{R}$ where $n \leq x$ for all $n \in \mathbb{N}$. Thus, $\mathbb{N}$ is bounded from above and hence, by the completeness axiom, there exists $u = \sup(\mathbb{N})$, with $u \in \mathbb{R}$.
Then $u-1$ is not a supremum of $\mathbb{N}$ by the def of supremum.
So there exists $m \in \mathbb{N}$ with $u-1 < m$.
But then $u < m+1$ and $m+1 \in \mathbb{N}$.
Contradiction. $\blacksquare$

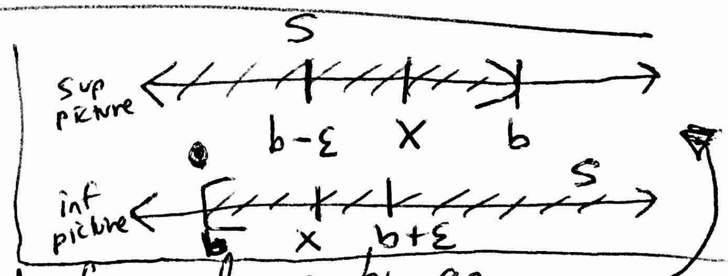
Ex: $x = 32.5632157$



Set $n = 33$

~~Note: If inf(S) and sup(S) exist then they are unique. HW~~

Theorem (Useful inf/sup fact)



Let $S \subseteq \mathbb{R}$ be nonempty.

(a) Suppose that S is bounded from above by an element b. Then, b is the supremum of S if and only if for every real number $\varepsilon > 0$ there exists $x \in S$ with $b - \varepsilon < x \leq b$.

(b) Suppose that S is bounded from below by an element b. Then, b is the infimum of S if and only if for every real number $\varepsilon > 0$ there exists $x \in S$ with $b \leq x < b + \varepsilon$.

8

proof of (a) (The proof of (b) is similar)

($\Longrightarrow$) Suppose that b is the supremum of S. Let $\varepsilon > 0$. Since $b - \varepsilon < b$ we know that $b-\varepsilon$ is not an upper bound for S. Thus there exists some $x \in S$ with $b - \varepsilon < x$. Since $x \in S$ and $b = \sup(S)$ we also have $x \le b$. Thus, $b - \varepsilon < x \le b$.

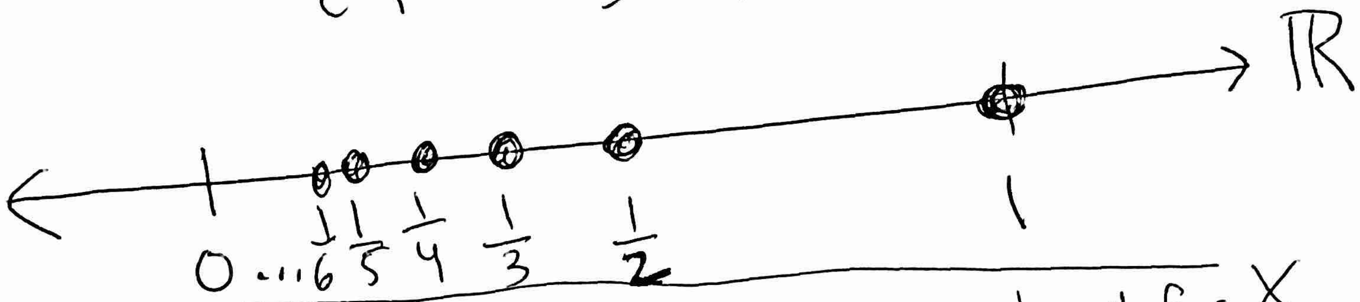
($\Longleftarrow$) Suppose that b is an upper bound for S and for every $\varepsilon > 0$ there exists $x \in S$ with $b - \varepsilon < x \le b$. Let's show that b is the supremum of S.

Pick any real number y with $y < b$. We want to show that y is <u>not</u> an upper bound for S. Let $\varepsilon = b - y > 0$. By our assumption there exists $x \in S$ with $b - \varepsilon < x \le b$. So, $y < x$. Thus, y is not an upper bound for S. ∎

9a

Ex: $X=\{\frac{1}{n} \mid n\in\mathbb{N}\}$

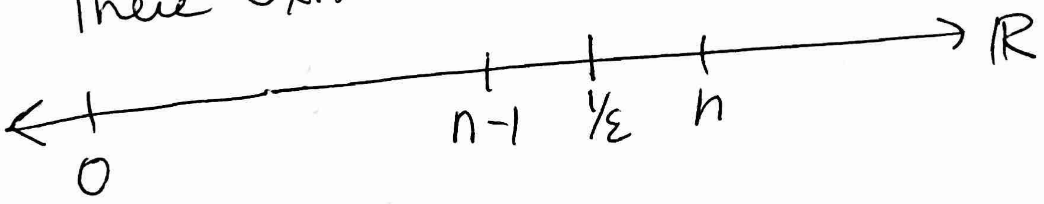
$=\{\frac{1}{1},\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\}$



Note that $\sup(X)=1$, since 1 is an upper bound for X and $1\in X$. (HW problem)

Claim: $\inf(X)=0$.

pf: Note that $0<\frac{1}{n}$ for all $n\in\mathbb{N}$, So, 0 is a lower bound for the set X. Let $\varepsilon>0$. By the Archimedean property, there exists $n\in\mathbb{N}$ such that $n>\frac{1}{\varepsilon}$.



Thus, $\frac{1}{n}<\varepsilon$.

Therefore, $0\le\frac{1}{n}<0+\varepsilon$.

By the useful inf/sup fact, $0=\inf(X)$.

$\blacksquare$

Ways to prove that a bound of a non-empty set is the infimum or supremum of the set

1. Use the useful sup/inf fact from page 7.

2. Let $S \subseteq \mathbb{R}$, $S \neq \phi$.
   - b is the supremum of S if
     (i) $x \leq b$ for all $x \in S$ (b is an upper bound of S)
     (ii) $b \leq c$ for all upper bounds c of S. (b is the least upper bound of S)
   - a is the infimum of S is
     (i) $a \leq x$ for all $x \in S$ (a is a lower bound of S)
     (ii) $c \leq a$ for all lower bounds c of S (a is the greatest lower bound of S)

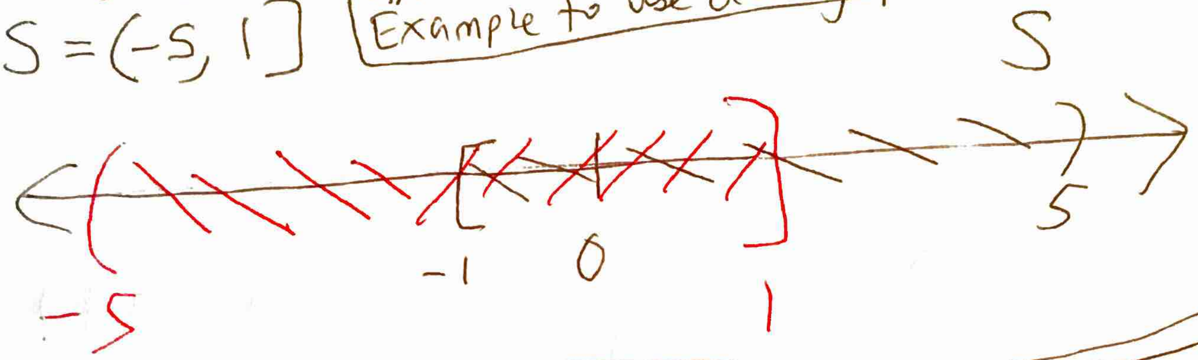
We now use the completeness axiom to prove this: 9c

Thm: If S is a non-empty subset of $\mathbb{R}$ that is bounded below, then the infimum of S exists as a real number.

$S = [-1, 5)$

$-S = (-5, 1]$

An Example to use during proof



pf: Suppose S is bounded from below and S is non-empty.

Let
$$-S = \{-x \mid x \in S\}.$$

We now show that $-S$ is bounded from above.

Let b be a lower bound for S.

Then $b \leq x$ for all $x \in S$.

So, $-b \geq -x$ for all $x \in S$,

Thus, $-b$ is an upper bound of $-S$.

By the completeness axiom $-S$ has a supremum. Let $b_{-S} = \sup(-S)$.

Then (1) $b_{-S} \geq -x$ for all $x \in S$ and

(2) $c \geq b_{-S}$ for all upper bounds of $-S$.

9d

Let $b_S = -b_{-S}$.

We claim that b_S is the infimum of S.

By (1), $b_{-S} \geq -x$ for all $x \in S$.

So, $b_S = -b_{-S} \leq x$ for all $x \in S$.

Hence

(1') b_S is a lower bound for S.

Now suppose that d is another lower bound for S. Then $d \leq x$ for all $x \in S$.

So, $-d \geq -x$ for all $x \in S$.

Hence $-d$ is an upper bound for $-S$.

Thus, by (2) we have that $-d \geq b_{-S}$.

Multiplying by -1 on both sides gives

(2') $d \leq -b_{-S} = b_S$ for all lower bounds d of S. Ie, b_S is the greatest lower bound of S

By (1') and (2'), $d = \inf(S)$.

∎

Well-ordering principle Every non-empty subset of $\mathbb{N}$ contains a least element.

Lemma: Let $y \in \mathbb{R}$, with $y > 0$. Then there exists $n \in \mathbb{N}$ with $n-1 \leq y < n$.

proof: Let $S = \{ m \in \mathbb{N} \mid y < m \}$.

By the Archimedean property, S is not empty. So, by the well-ordering principle, there exists $n_0 \in S$ with $n_0 \leq m$, $\forall m \in S$. So, $n_0 - 1 \notin S$.
Then $n_0 - 1 \leq y < n_0$. $\blacksquare$

Density Theorem: Let $x, y \in \mathbb{R}$ and $x < y$. Then there exists $r \in \mathbb{Q}$ with $x < r < y$.

proof:
We first prove the theorem for $x > 0$.
Since $x < y$ we have $0 < y - x$. So, $0 < \frac{1}{y-x}$.
By the Archimedean property, there exists $n \in \mathbb{N}$ with $0 < \frac{1}{y-x} < n$.

Thus, $\frac{1}{n} < y - x$.

So, $1 < ny - nx$ and so, $1 + nx < ny$.

Apply the lemma to $nx > 0$ to obtain $m \in \mathbb{N}$ with $m - 1 \leq nx < m$.

So, $m \leq nx + 1 < ny$.

by adding 1 to $m-1 \leq nx$

So, $nx < m < ny$.

So, $x < \frac{m}{n} < y$.

Set $r = \frac{m}{n}$.

What about if $x < 0$?

Let $s \in \mathbb{N}$ with $0 < x + s$.

Apply the above to $x + s < y + s$ to obtain $r \in \mathbb{Q}$ with $x + s < r < y + s$.

Then $x < r - s < y$ and $r - s \in \mathbb{Q}$. $\blacksquare$

Absolute Value

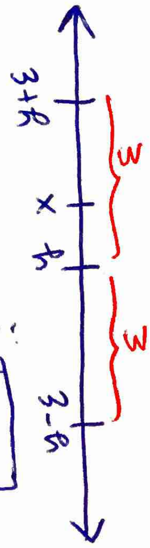
Def: Let $x \in \mathbb{R}$. Define the absolute value of x to be

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Ex: $|3.72| = 3.72$

$|-5| = -(-5) = 5$

The absolute value is our method to measure distances in $\mathbb{R}$.

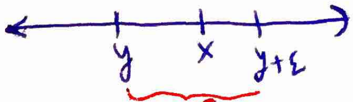


Suppose we have $x, y, \varepsilon \in \mathbb{R}$ with $\varepsilon > 0$ and $|x-y| < \varepsilon$.

case 1: $(x-y) \geq 0$

Then, $0 < \underbrace{x-y}_{|x-y|} < \varepsilon$

So, $\boxed{y < x < y+\varepsilon}$



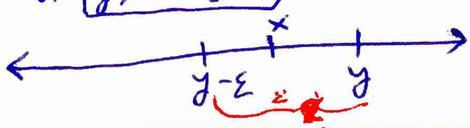
case 2: $(x-y) < 0$

$0 < \underbrace{-(x-y)}_{|x-y|} < \varepsilon$

So, $0 < -x+y < \varepsilon$

or $0 > x-y > -\varepsilon$

or $\boxed{y > x > y-\varepsilon}$



In either case $y-\varepsilon < x < y+\varepsilon$.

In fact

$\boxed{|x-y| < \varepsilon \text{ iff } y-\varepsilon < x < y+\varepsilon}$

That is, $|x-y| < \varepsilon$ means x and y are within ε of each other.

Facts about absolute value:

Let $a, b \in \mathbb{R}$. Then:

1. $|ab| = |a| \cdot |b|$
2. Let $c > 0$. Then $|a| \le c$ if and only if $-c \le a \le c$.
3. $-|a| \le a \le |a|$
4. $|a+b| \le |a| + |b|$ (The triangle inequality)
5. $\big||a| - |b|\big| \le |a-b|$
6. $|a-b| \le |a| + |b|$

proof of 2:

Let $c > 0$.

($\Rightarrow$) Suppose $|a| \le c$.

If $a < 0$ then $\underbrace{a < -a}_{a<0} \underbrace{\le c}_{|a| \le c}$

If $a > 0$ then $\underbrace{-a \le a}_{a>0} \underbrace{\le c}_{|a| \le c.}$

So, $a \le c$ and $-a \le c$.

So, $-c \le a \le c$.

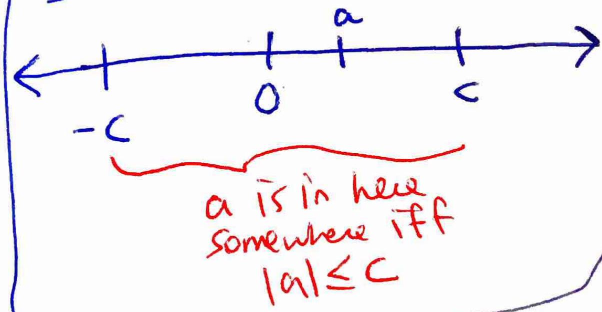
($\Leftarrow$) Suppose $-c \le a \le c$.

Then, $-c \le a$ and $a \le c$.

So, $-a \le c$ and $a \le c$.

So, $|a| \le c$. $\boxed{2}$

PICTURE FOR 2



Pf of 3

Take $c = |a|$ in part 2. $\boxed{3}$

Pf of 4:

proof of 4:

From 3 we know $-|a| \le a \le |a|$ and $-|b| \le b \le |b|$. Adding these inequalities gives

$$-(|a|+|b|) \le a+b \le |a|+|b|.$$

Thus by 2 we have that

$$|a+b| \le |a| + |b|.$$

$\blacksquare$

~~Corollary~~