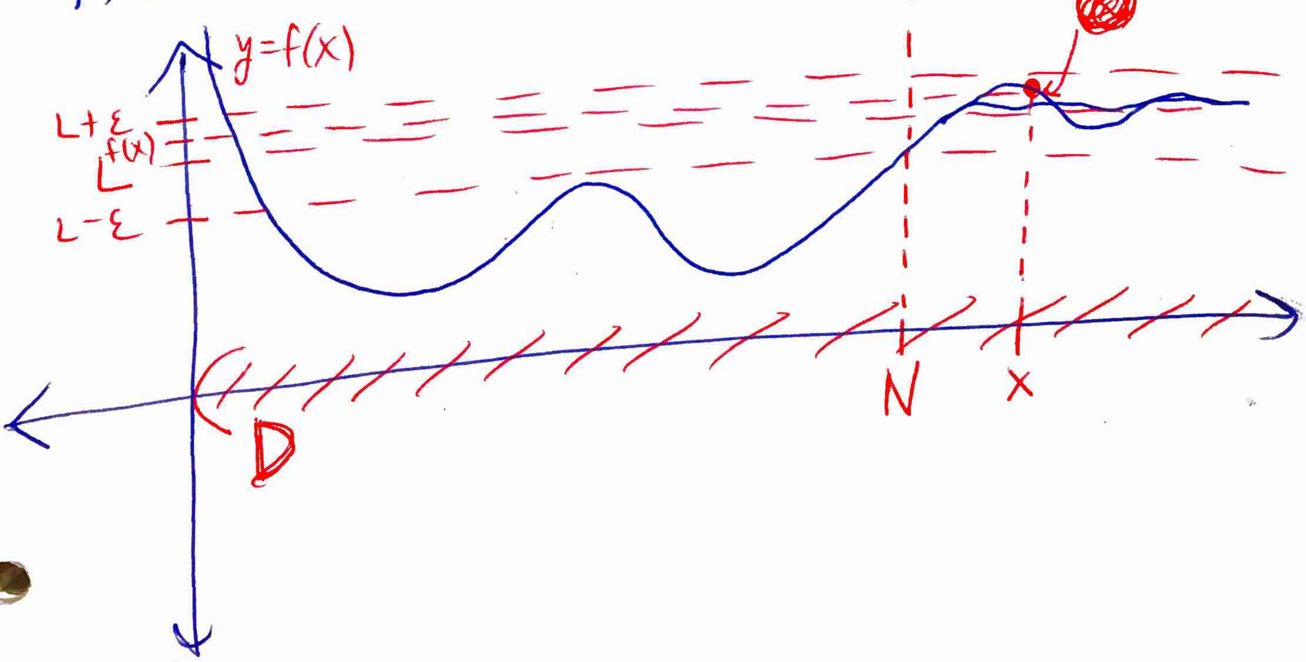
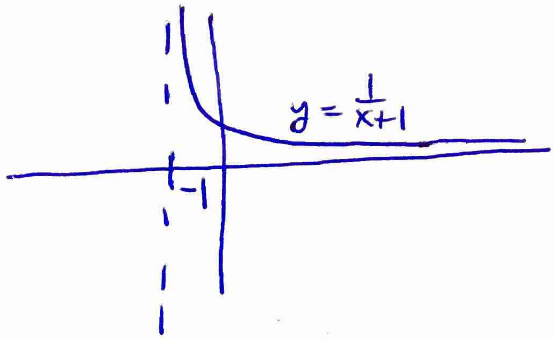
Limits of functions at infinity

Def: Let f be a real-valued function defined on some set D containing an ~~some~~ interval of the form (a,∞). Let $L \in \mathbb{R}$. We say that ~~the limit of f~~ the limit of f as x tends to infinity is equal to L if for every $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that if $x \in D$ and $x > N$, then $|f(x) - L| < \varepsilon$.



Ex: $f(x) = \frac{1}{x+1}$



$D = \mathbb{R} \setminus \{-1\}$

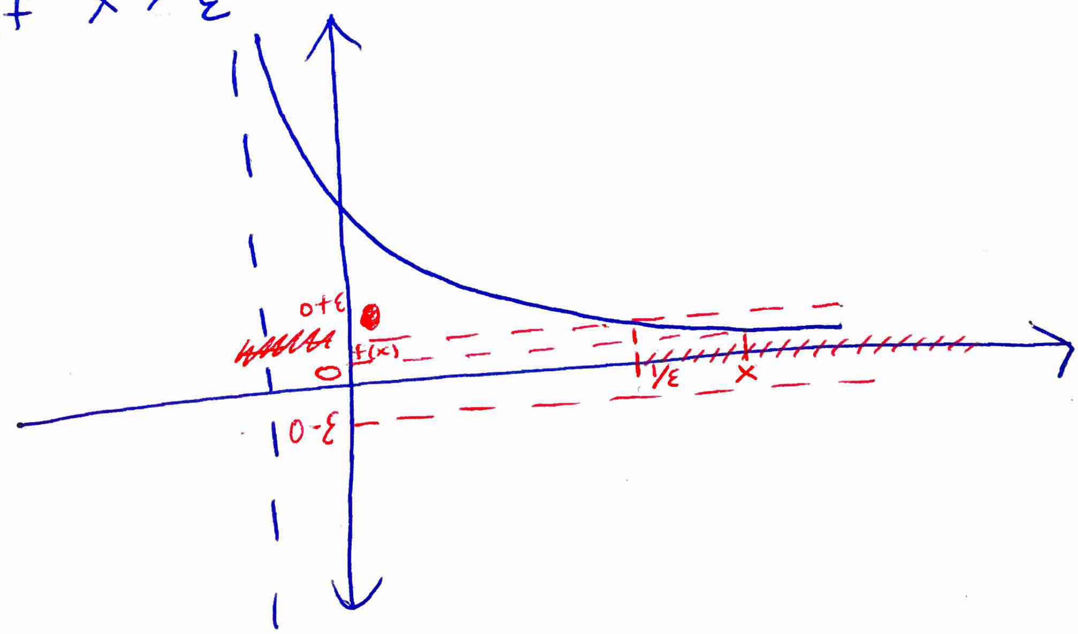
$$\lim_{x \to \infty} f(x) = 0$$

Let $\varepsilon > 0$.

If $x > -1$, then $|f(x) - 0| = \left|\frac{1}{x+1} - 0\right| = \left|\frac{1}{x+1}\right| = \frac{1}{x+1}$.

Let $N = \frac{1}{\varepsilon}$.

If $x > \frac{1}{\varepsilon}$, then $|f(x) - 0| = \frac{1}{x+1} < \frac{1}{x} < \varepsilon$. ∎



Ex: Suppose $D \subseteq \mathbb{R}$ containing some interval (a,∞). Suppose $f, g : D \to \mathbb{R}$. Let $L, M \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$.

If $\lim\limits_{x\to\infty} f(x) = L$ and $\lim\limits_{x\to\infty} g(x) = M$,

then

$$\lim_{x\to\infty} f(x) + g(x) = L \pm M = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x).$$

Pf: Let $\varepsilon > 0$.

There exists N_1 such that if $x \in D$ and $x > N_1$, then $|f(x) - L| < \frac{\varepsilon}{2}$.

There exists N_2 such that if $x \in D$ and $x > N_2$, then $|g(x) - M| < \frac{\varepsilon}{2}$.

If $N = \max\{N_1, N_2\}$ and $x \in D$ with $x > N$, then

$$\begin{aligned} |f(x) + g(x) - (L+M)| &= |f(x) - L + g(x) - M| \\ &\le |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Limit of a Function at $x=a$

or cluster point
or accumulation point

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Def: Let $D\subseteq\mathbb{R}$, Let $a\in\mathbb{R}$. We say that a is a ~~[illegible]~~ limit point of D if for every $\delta>0$ there exists $x\in D$ with $x\neq a$ and $|x-a|<\delta$.

Ex: $D=(0,1]$

0 is a ~~cluster~~ limit point.

1 is a ~~cluster~~ limit point

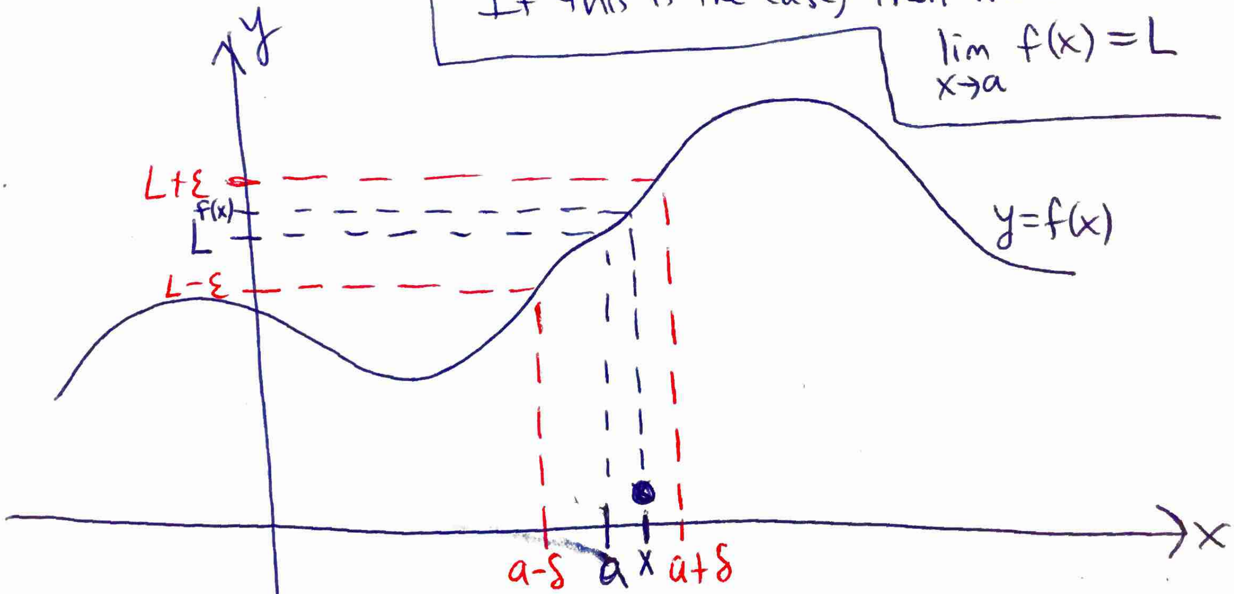
2 is not a ~~cluster~~ limit point

Ex: 0 is a limit point of $\{\frac{1}{n} \mid n\in\mathbb{Z}\}$.

Def: Let $D\subseteq\mathbb{R}$, $f:D\to\mathbb{R}$. Let a be a limit point of D. We say that f has a limit as x tends to a if there exists $L\in\mathbb{R}$ such that for every $\varepsilon>0$ there exists a $\delta>0$ such that for every $x\in D$ we have that $0<|x-a|<\delta$ implies $|f(x)-L|<\varepsilon$.

If this is the case, then we write

$$\lim_{x\to a} f(x)=L$$



Note: $0<|x-a|$ rules out $x=a$ in the def.

We want x close to a but not equal to a.

Ex: Let $f(x)=x^2$.
Let's show $\lim_{x\to 2} x^2 = 4$ using the ε-δ def of limit.

Let $\varepsilon > 0$. Need to find $\delta > 0$ so that if $|x-2|<\delta$, then $|x^2-4|<\varepsilon$.

Note that $|x^2-4| = \underbrace{|x-2|}_{\text{This is the part that will be less than } \delta.}\underbrace{|x+2|}_{\text{First Goal: bound this part and make it dissapear by picking } \delta \text{ small enough}}$

Suppose $\delta \leq 1$.
If $|x-2|<\delta\leq 1$, then $|x+2| = |x-2+2+2|$
$\leq |x-2|+|4| < 1+4 = 5$.

Thus, if $|x-2|<\delta\leq 1$, then $|x^2-4| = |x-2||x+2| < 5|x-2|$.

Let $\delta = \min\{1, \frac{\varepsilon}{5}\}$.

Then if $|x-2|<\delta$ we have that

$$|x^2-4| = |x-2||x+2| < 5|x-2| \leq 5\cdot\frac{\varepsilon}{5} = \varepsilon. \quad \blacksquare$$

Ex: Let's show that $\lim_{x\to -3} \frac{1}{x+2} = -1$.

Let $\varepsilon > 0$. $\boxed{\text{We need to find } \delta > 0 \text{ so that if } |x-(-3)| < \delta \text{ then } \left|\frac{1}{x+2} - (-1)\right| < \varepsilon.}$

Note that

$$\left|\frac{1}{x+2} - (-1)\right| = \left|\frac{1+x+2}{x+2}\right| = \left|\frac{x-(-3)}{x+2}\right| = \underbrace{|x-(-3)|}_{\text{This is the part that will be less than } \delta} \underbrace{\left|\frac{1}{x+2}\right|}_{\text{Lets make } \delta \text{ small so we can bound this.}}$$

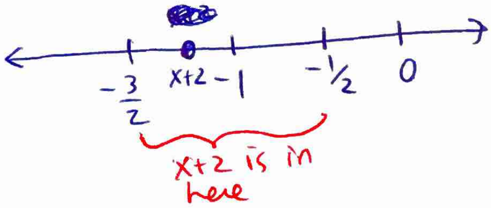
Suppose $\delta \le \frac{1}{2}$.

If $0 < |x-(-3)| < \delta \le \frac{1}{2}$,

then $|x+3| < \frac{1}{2}$,

so $-\frac{1}{2} < x+3 < \frac{1}{2}$,

so $-\frac{3}{2} < x+2 < -\frac{1}{2}$, $\leftarrow$



so, $|x+2| > \frac{1}{2}$,

so, $\frac{1}{|x+2|} < 2$.

Set $\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{2}\right\}$.

If $|x-(-3)| < \delta$, then

$$\left|\frac{1}{x+2} - (-1)\right| = |x-(-3)|\left|\frac{1}{x+2}\right| < 2|x-(-3)| < 2\cdot\frac{\varepsilon}{2} = \varepsilon.$$

$\blacksquare$

Thm: Let $A \subseteq \mathbb{R}$ and $f, g : A \to \mathbb{R}$.
Let a be a ~~cluster~~ limit point of A. Let $\alpha \in \mathbb{R}$.
Then

① If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

(a) $\lim_{x\to a} \alpha = \alpha$

(b) $\lim_{x\to a} (f(x) + g(x)) = L + M$

(c) $\lim_{x\to a} (f(x) - g(x)) = L - M$

(d) $\lim_{x\to a} f(x)g(x) = LM$

(e) $\lim_{x\to a} \alpha f(x) = \alpha L$

② If $h : A \to \mathbb{R}$, $h(x) \neq 0$ for all $x \in A$, and $\lim_{x\to a} h(x) = H \neq 0$
then $\lim_{x\to a} \frac{f(x)}{h(x)} = \frac{L}{H}$.

① (a) Let $\varepsilon > 0$. Pick any $\delta > 0$. If $0 < |x - a| < \delta$
Then $|\alpha - \alpha| = 0 < \varepsilon$.

Pf: ① (b) Let $\varepsilon > 0$.
Since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ so that if $x \in A$ and $0 < |x - a| < \delta_1$, then $|f(x) - L| < \frac{\varepsilon}{2}$.
Since $\lim_{x\to a} g(x) = L$, there exists $\delta_2 > 0$ so that if $x \in A$ and $0 < |x - a| < \delta_2$, then $|g(x) - M| < \frac{\varepsilon}{2}$.
Let $\delta = \min\{\delta_1, \delta_2\}$.
If $x \in A$ and $0 < |x - a| < \delta$, then

$$\begin{aligned} |f(x) + g(x) - (L + M)| &= |f(x) - L + g(x) - M| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

① (d) Let $\varepsilon > 0$. If $\alpha = 0$, Then $\lim_{x \to a} \alpha f(x) = \lim_{x \to a} 0 = 0$ by ① (a). Suppose $\alpha \neq 0$.

Since $\lim_{x \to a} f(x) = L$, There exists $\delta > 0$ so that if $x \in A$ and $0 < |x - a| < \delta$ we have that $|f(x) - L| < \frac{\varepsilon}{|\alpha|}$. 38

Then, if $x \in A$ and $0 < |x - a| < \delta$ we have that

$$|\alpha f(x) - \alpha L| = |\alpha| |f(x) - L| < |\alpha| \frac{\varepsilon}{|\alpha|} = \varepsilon.$$

~~[illegible]~~

~~② See next page~~

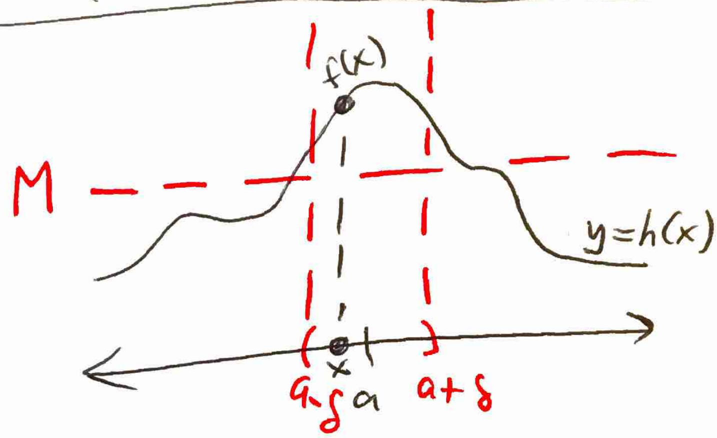
② We will prove that $\lim_{x \to a} \frac{1}{h(x)} = \frac{1}{H}$.

Then by part 1 (d) which is a HW problem we will have that

$$\lim_{x \to a} \frac{f(x)}{h(x)} = \lim_{x \to a} f(x) \cdot \frac{1}{h(x)} = L \cdot \frac{1}{H} = \frac{L}{H}.$$

We need a lemma:

Suppose that $\lim_{x \to a} h(x) = H \neq 0$. Then there exists $\delta > 0$ and $M > 0$ where if $0 < |x-a| < \delta$ and $x \in A$, then $|h(x)| > M$.



pf of lemma:

Let $\varepsilon = \frac{|H|}{2} > 0$.

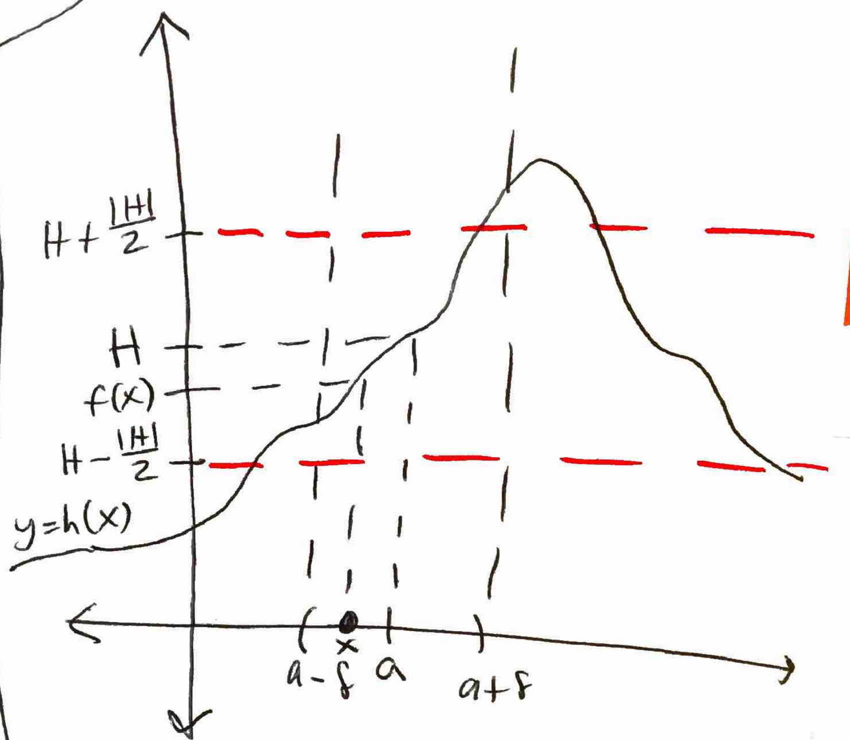
Since $\lim_{x \to a} h(x) = H$ there exists $\delta > 0$ where if $x \in A$ and $0 < |x-a| < \delta$ then $|h(x) - H| < \frac{|H|}{2}$.

That is,

$$\begin{aligned} |H| &= |H - h(x) + h(x)| \\ &\leq |H - h(x)| + |h(x)| \\ &< \frac{|H|}{2} + |h(x)|. \end{aligned}$$

So, if $x \in A$ and $0 < |x-a| < \delta$ then, $\frac{|H|}{2} < |h(x)|$.

Set $M = \frac{|H|}{2}$.



Now we show that $\lim\limits_{x \to a} \frac{1}{h(x)} = \frac{1}{H}$.

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Let $\varepsilon > 0$.

Note that

$$\left| \frac{1}{h(x)} - \frac{1}{H} \right| = \left| \frac{H - h(x)}{h(x) \cdot H} \right| = \frac{|h(x) - H|}{|h(x)|\,|H|}$$

By the lemma there exists $\delta_1 > 0$ and $M > 0$ so that if $x \in A$ and $0 < |x-a| < \delta_1$ then $|h(x)| > M$.

Since $\lim\limits_{x \to a} h(x) = H$ there exists δ_2 so that if $x \in A$ and $0 < |x-a| < \delta_2$ then $|h(x) - H| < \varepsilon\,(M \cdot |H|)$.

Let $\delta = \min\{\delta_1, \delta_2\}$. then If $x \in A$ and $0 < |x-a| < \delta$

$$\left| \frac{1}{h(x)} - \frac{1}{H} \right| = \frac{|h(x) - H|}{|h(x)|\,|H|} < \frac{|h(x) - H|}{M \cdot |H|}$$

since $|h(x)| > M$

$$< \frac{M \cdot |H| \cdot \varepsilon}{M \cdot |H|} = \varepsilon. \quad \boxed{\diagup\diagup}$$

Since $|h(x) - H| < (\varepsilon \cdot M \cdot |H|)$