Compactness

Def: Let $S \subseteq \mathbb{R}$. An <u>open cover</u> of S is a collection $X = \{ \mathcal{O}_\alpha \}$ of open sets such that $S \subseteq \bigcup_\alpha \mathcal{O}_\alpha$.

If $X' \subseteq X$ such that $S \subseteq \bigcup_{\mathcal{O}_\alpha \in X'} \mathcal{O}_\alpha$ then X' is called a <u>subcover</u> of X. If in addition, X' is finite then it is called a <u>finite subcover</u>.

Ex: Let $S = [1, \infty)$.

$\{ (0, \infty) \}$ is an open cover of S.

$\{ (n-1, n+1) \mid n \in \mathbb{N} \}$ is an open cover of S.

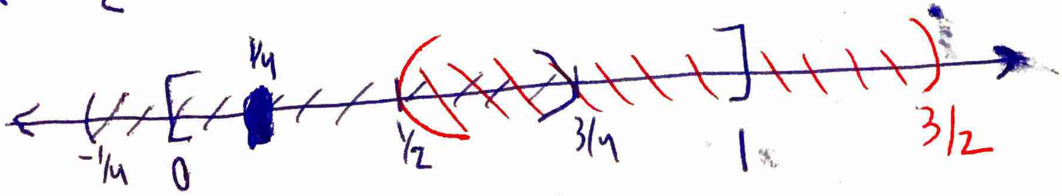
$\{ (0, n) \mid n \in \mathbb{N} \}$ is an open cover of S.

Ex: $S = [0, 1]$

$$X = \left\{ \left(\tfrac{1}{n} - \tfrac{1}{2}, \tfrac{1}{n} + \tfrac{1}{2} \right) \mid n \in \mathbb{N} \right\}$$

X is a cover of S.

$$X' = \left\{ \left(\tfrac{1}{4} - \tfrac{1}{2}, \tfrac{1}{4} + \tfrac{1}{2} \right), \left(1 - \tfrac{1}{2}, 1 + \tfrac{1}{2} \right) \right\}$$ is a finite subcover.



Def: Let $S \subseteq \mathbb{R}$. We say that S is compact if every open cover of S contains a finite subcover.

Ex: Let $S = [0, \infty)$.

We show that S is not compact. That is, we find an open cover of S that does not have a finite subcover.

Let $X = \{(-1, n) \mid n \in \mathbb{N}\}$

$= \{(-1,1), (-1,2), (-1,3), (-1,4), \ldots\}$

Suppose that $X' = \{(-1,n_1), (-1,n_2), \ldots, (-1,n_k)\}$ is a finite subset of X. We show X' cannot cover S.

Let $m = \max\{n_1, n_2, \ldots, n_k\}$.

Then $m+1 \in S$, but $m+1 \notin \bigcup_{i=1}^{k} (-1, n_i)$.

~~**Def:** Let $S \subseteq \mathbb{R}$. We say that S is bounded if there exists $M > 0$ such that $S \subseteq [-M, M]$.~~

~~Ex: $(0,1)$ is bounded~~

~~$[0, \infty)$ is not bounded~~

<u>Def:</u> Let $S \subseteq \mathbb{R}$. We say that S is <u>bounded</u> if there exists $M > 0$ such that $S \subseteq [-M, M]$.

<u>Ex:</u> $[0, 3)$ is bounded.
$[0, \infty)$ is not bounded.

OTHER DIRECTION IS A HANDOUT ~~AFTER~~ THIS

<u>Heine-Borel</u>: Let $K \subseteq \mathbb{R}$.
K is compact if and only if K is closed and bounded.

(we only prove this direction)

$(\Leftarrow)$ Suppose that K is closed and bounded.
Let $G = \{G_\alpha\}$ be an open covering of K.
We want to show that K is contained in some finite subcover of G.
We prove this by contradiction.
Suppose that K is not contained in any finite subcover of G. ~~in the union of any finite number of sets in G~~.

By hypothesis, K is bounded.
So, $K \subseteq [-r, r]$ for some $r \in \mathbb{R}$ with $r > 0$.
Let $I_1 = [-r, r]$.
Bisect I_1 into two intervals $I_1' = [-r, 0]$ and $I_1'' = [0, r]$
At least one of $K \cap I_1'$ or $K \cap I_1''$ is nonempty and has the property that it is not contained

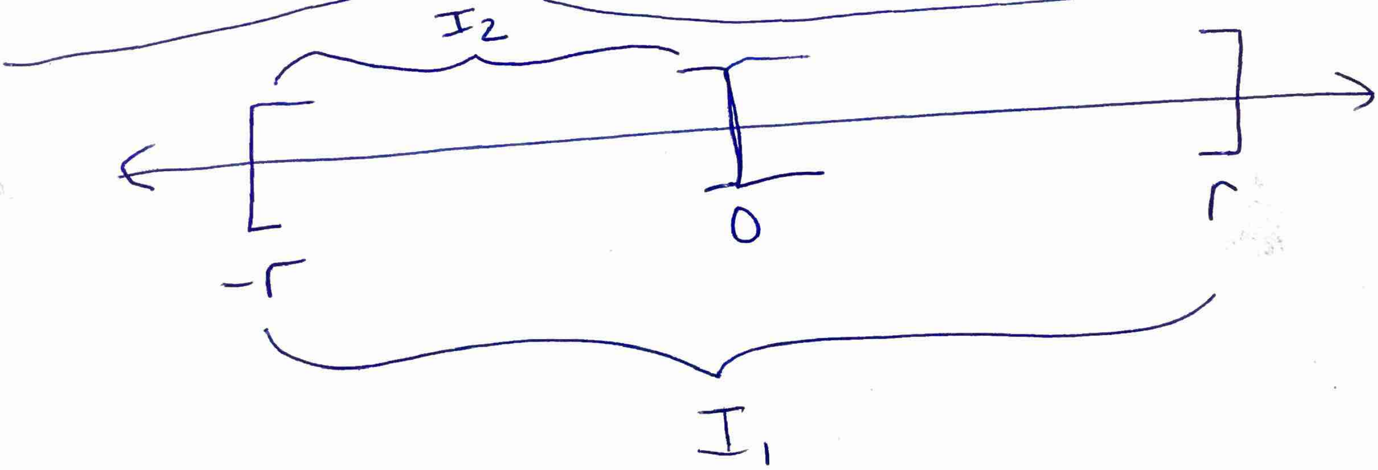
a finite subcover of G.

in ~~the union of any finite number of sets from~~ G,

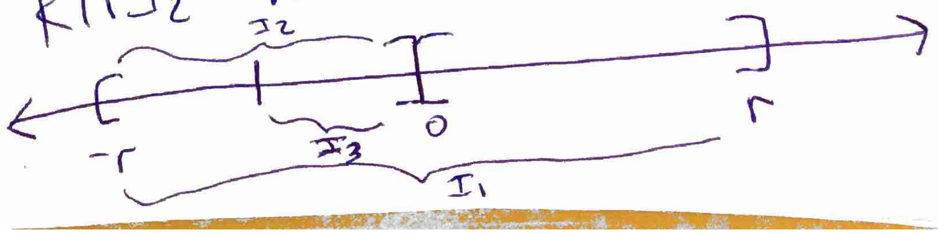
[For if both of the sets $K \cap I_1'$ and $K \cap I_1''$ are contained in ~~the union of some~~ a finite ~~number of sets~~ subcover of ~~in~~ G, then $K = (K \cap I_1') \cup (K \cap I_1'')$ is contained in ~~the union of some~~ a finite ~~number~~ subcover of ~~sets in~~ G.]

let $I_2 = I_1'$ if $K \cap I_1'$ is not contained in the union of some finite number of sets in G,

otherwise set, $I_2 = I_1''$ $K \cap I_1''$ has this property,



We now bisect I_2 into two closed subintervals I_2' and I_2''. If $K \cap I_2'$ is nonempty and is not contained in the union of some finite number of sets in G we let $I_3 = I_2'$. if $K \cap I_2''$ has this property, otherwise set $I_3 = I_2''$.



Continuing this process we get a nested sequence of intervals:

$[-r,r]=I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \ldots$

where $K\cap I_i \neq \emptyset$ and $K \cap I_i$ can't be covered by a finite subcover of X.

Claim: There is a point $\xi \in \mathbb{R}$ that belongs to all the intervals. That is, $\xi \in \bigcap_{k=1}^{\infty} I_k$.

pf: Suppose that $I_n = [a_n, b_n]$.

Since the intervals are nested we have that $a_n \le b_1$ for all n. So the sequence (a_n) is bounded above.

Let $\xi = \sup\{a_n \mid n \in \mathbb{N}\}$.

So, $a_n \le \xi$ for all n.

We claim that $\xi \le b_n$ for all n.

This is established by showing that for any particular n, b_n is an upper bound of $\{a_k \mid k \in \mathbb{N}\}$.

We consider two cases. Let n be fixed.

case (i) If $n \le k$, then since $I_k \subseteq I_n$ we have $a_n \le a_k \le b_k \le b_n$. So, $a_k \le b_n$.

case (ii) If $k < n$, then $I_n \subseteq I_k$.

So, $a_k \le a_n \le b_n \le b_k$. So, $a_k \le b_n$.

Therefore, $a_n \le \xi \le b_n$ for all n.

Hence $\xi \in \bigcap I_k$. $\boxed{/\!/}$ (claim)

Note that each I_n contains infinitely many points from K. [For if $I_n \cap K$ was finite then there would be a subcover of it. See hw: finite sets are compact.]

Thus, ξ is a limit point of K [since there is a sequence of points of K converging to ξ.] see next pages 55′ 55″

Since K is closed, we know that $\xi \in K$.

Therefore there is some open set G_ξ from G such that $\xi \in G_\xi$ [because the sets from G cover K.]

Since G_ξ is open, there exists $\varepsilon > 0$ so that

$$(\xi - \varepsilon, \xi + \varepsilon) \subseteq G_\xi .$$

Note that

length of $I_1 = 2r$

length of $I_2 = r$

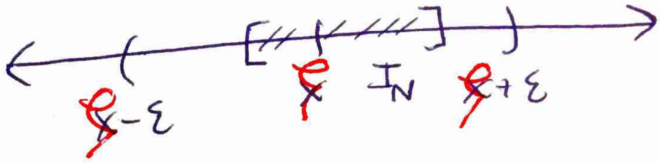
length of $I_3 = \frac{r}{2}$

length of $I_4 = \frac{r}{2^2}$

$\vdots$

length of $I_n = \frac{r}{2^{n-2}}$

Since r is fixed, the sequence $\frac{r}{2^{n-2}}$ converges to 0. So, there exists N so that $\frac{r}{2^{N-2}} < \varepsilon$. That is, $I_N \subseteq (\xi-\varepsilon, \xi+\varepsilon)$.



So, $K \cap I_N \subseteq (\xi-\varepsilon, \xi+\varepsilon) \subseteq G_\xi$.

But then $K \cap I_N$ would be covered by a single element of G. This is a contradiction! $\blacksquare$

ρ is a limit point of K

Let $\varepsilon > 0$.

Since $\rho = \sup\{a_k \mid k \in \mathbb{N}\}$

then there exists $k_1 \in \mathbb{N}$ where

$\rho - \varepsilon < a_{k_1} \leq \rho$.

One can show that

$\rho = \inf\{b_k \mid k \in \mathbb{N}\}$ (see proof on next page).

Hence there exists $k_2 \in \mathbb{N}$

where

$\rho \leq b_{k_2} < \rho + \varepsilon$.

Let $k = \max\{k_1, k_2\}$.

Then $\boxed{\rho - \varepsilon} < a_{k_1} \leq a_k \leq \rho \leq b_k \leq b_{k_2} < \rho + \varepsilon$.

Hence, $I_k = [a_k, b_k] \subseteq (\rho - \varepsilon, \rho + \varepsilon)$.

~~We know that $K \cap I_k \neq \emptyset$.~~ ~~There exists a point of~~ I_k contains infinitely many points from K (from proof). So, $(\rho - \varepsilon, \rho + \varepsilon)$ contains a point from K that isn't ρ. ▨

55''

Claim:

Let's show that $\rho = \inf\{b_k \mid k \in \mathbb{N}\}$
$= \inf\{b_1, b_2, b_3, b_4, \ldots\}$

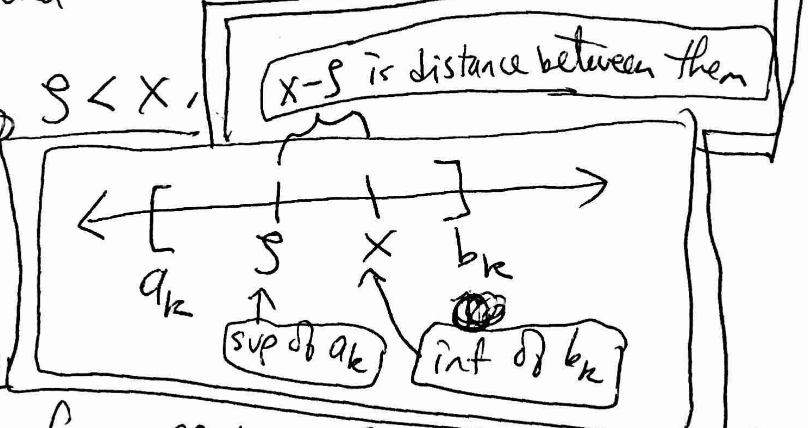
Let $x = \inf\{b_k \mid k \in \mathbb{N}\}$.

We saw (in the proof) that $\rho \le b_k$ for all k. So, ρ is a lower bound of the set $\{b_k \mid k \in \mathbb{N}\}$

Since x is the $\underbrace{\text{infimum}}_{\text{greatest lower bound}}$ of $\{b_k \mid k \in \mathbb{N}\}$

then $\rho \le x$.

Suppose that $\rho < x$. We show this leads to a contradiction.



Then we have that

$a_k \le \rho < x \le b_k$ for all k. See picture

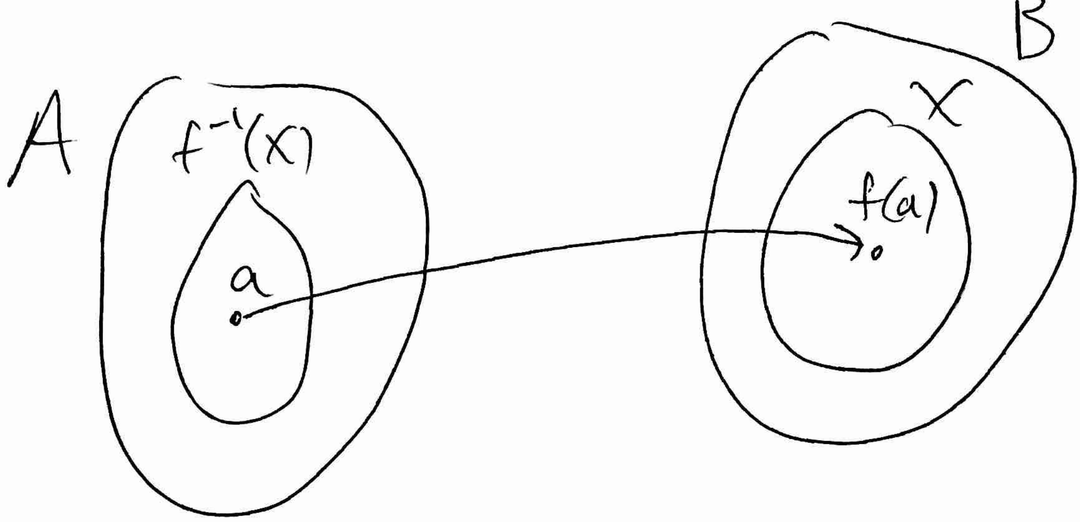
But we know that, by construction, the length of $[a_k, b_k] = \dfrac{r}{2^{k-2}}$.

Since $\dfrac{r}{2^{k-2}} \to 0$ as $k \to \infty$ we can make $\dfrac{r}{2^{k-2}} < x - \rho$.
For this to happen we would need either $\rho < a_k$ or $b_k < x$ which can't happen. Thus, $\rho = x$.

$\boxed{\text{Claim}}$

Applications of Compactness

Def: Let $f : A \to B$ where A and B are sets. Let $X \subseteq B$.
Then $f^{-1}(X) = \{a \in A \mid f(a) \in X\}$



Thm: Let $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$ is open. Let $\Theta \subseteq \mathbb{R}$ be an open set. If f is continuous on D, then $f^{-1}(\Theta)$ is open. 58'

proof: Let $\Theta \subseteq \mathbb{R}$ be open. We want to show that $f^{-1}(\Theta)$ is open.

Let $a \in f^{-1}(\Theta)$. Therefore $f(a) \in \Theta$.

Since Θ is open there exists $\varepsilon > 0$ where $(f(a)-\varepsilon, f(a)+\varepsilon) \subseteq \Theta$.

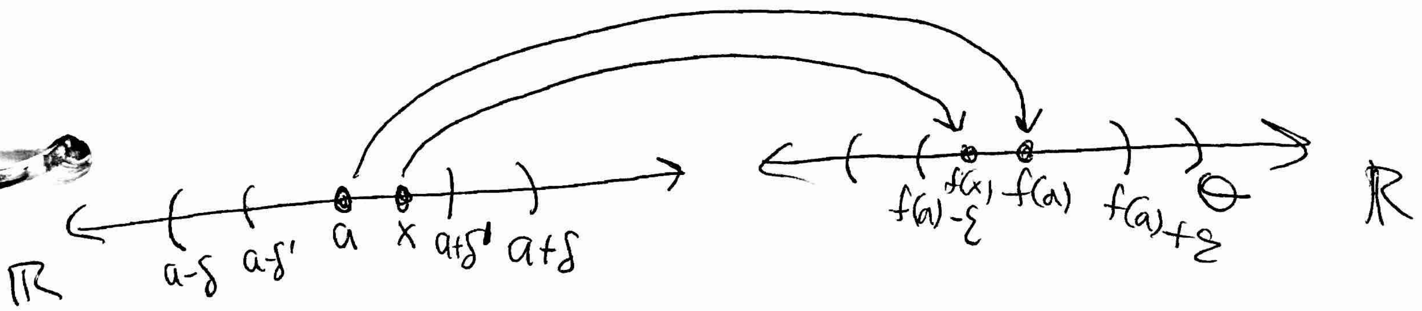
Since f is continuous at a, there exists $\delta > 0$ where if $x \in D$ and $|x-a| < \delta$ then $|f(x)-f(a)| < \varepsilon$.

Since D is open and $a \in D$ there exists δ' where $\delta' < \delta$ and $(a-\delta', a+\delta') \subseteq D$.

Then if $\underbrace{|x-a| < \delta'}_{x \in (a-\delta', a+\delta')}$, then $x \in D$ and

$$\underbrace{f(x) \in (f(a)-\varepsilon, f(a)+\varepsilon)}_{|f(x)-f(a)|<\varepsilon} \subseteq \Theta$$

So, $(a-\delta', a+\delta') \subseteq f^{-1}(\Theta)$.



(D is open)

Application: Let $f: D \to \mathbb{R}$ be continuous.
If $X \subseteq D$ is closed and bounded (compact), then $f(X)$ is closed and bounded (compact).

proof: Suppose $X \subseteq D$ is compact.
Consider $f(X) = \{ f(a) \mid a \in X \}$.
Let $G = \{ G_\alpha \}$ be an open cover of $f(X)$.

Consider $G' = \{ f^{-1}(G_\alpha) \}$.

Then G' is an open cover of X:
① $f^{-1}(G_\alpha)$ is open since G_α is open
② If $a \in X$, then $f(a) \in f(X)$. So, $f(a) \in G_{\alpha a}$ for some α_a. Then $a \in f^{-1}(G_{\alpha a})$.

So, there exists a finite subcover $\{ f^{-1}(G_{\alpha_1}), f^{-1}(G_{\alpha_2}), \ldots, f^{-1}(G_{\alpha_n}) \}$ that covers X.

Thus, $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}$ is a finite subcover of $f(X)$
[since if $f(a) \in f(X)$ for some $a \in X$, then $a \in f^{-1}(G_{\alpha_i})$ for some i and thus, $f(a) \in G_{\alpha_i}$]

∎

Corollary: Suppose $f: D \to \mathbb{R}$ where D is open. Let f be continuous on D. Let $X \subseteq D$ where X is compact (closed/bounded). Then there exists $a, b \in X$ where $f(a) \leq f(x)$ $\forall x \in X$ and $f(x) \leq f(b)$ $\forall x \in X$.

pf: By the previous thm, $f(X)$ is compact. So, $f(X)$ is bounded.
Let $\hat{a} = \inf(f(X))$ and $\hat{b} = \sup(f(X))$.

Hw problem: Since $f(X)$ is closed, $\hat{a}, \hat{b} \in f(X)$.
So there exists $a, b \in X$ where $f(a) = \hat{a}$ and $f(b) = \hat{b}$. Then, $f(a) \leq f(x)$ $\forall x \in X$ and $f(b) \geq f(x)$ $\forall x \in X$. $\blacksquare$